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ON THE EVOLUTION OPERATOR FOR ABSTRACT PARABOLIC EQUATIONS

BY

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ABSTRACT

We find a new construction of the evolution operator $G(t,s)$ associated to a family $\{A(t),\ 0 \leqq t \leqq T\}$ of generators of analytic semigroups in a Banach space X. We study the dependence of $G(t,s)$ on t and s, and we give regularity results for the solution of the i.v.p. $u'(t) = A(t)u(t) + f(t)$, $u(0) = x$.

0. Introduction

The purpose of this paper is a new construction of a parabolic evolution operator $G(t,s)$ in general Banach space X, satisfying

$$(0.1) \qquad \begin{cases} G_t(t,s) = A(t)G(t,s), & t > s, \\ G(s,s) = I. \end{cases}$$

Here the linear operators $A(t) : D(A(t)) \subset X \rightarrow X$ generate analytic semigroups $e^{\sigma A(t)}$ in X and have constant domain $D(A(t)) = D$; the function $t \rightarrow A(t)$ is assumed to be α-Hölder continuous with values in $L(D, X)$. D is not necessarily dense in X; this is the unique difference between our hypotheses and the classical Sobolevskiĭ–Tanabe ones (see [S1], [T]).

The technique used to construct $G(t,s)$ is very similar to the one employed in [DPS], [LS]: that is, we consider $A(t)$ as a perturbation of $A(s)$ and we use new maximal regularity results for the time-independent case. More precisely, for any $x \in X$ we set

$$(0.2) \qquad G(t,s)x = e^{(t-s)A(s)}x + W(t,s)x$$

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where $w = W(\cdot, s)x$ is the solution of

$$
(0.3) \qquad \begin{cases} w'(t) = A(s)w(t) + [A(t) - A(s)](w(t) + e^{(t-s)A(s)}x), & t > s, \\ w(s) = 0. \end{cases}
$$

For general $x \in X$, the function $t \to [A(t) - A(s)]e^{(t-s)A(s)}x$ is α-Hölder continuous in $[s + \delta_1, s + \delta_2]$ for $0 < \delta_1 < \delta_2$, and it has a singularity like $(t - s)^{\alpha - 1}$ for t near s. Therefore, to find a solution of (0.3), we have to work in a space Y of X-valued functions having a singularity at $t = s$, but being sufficiently regular for $t > s$. We need also that Y has the so-called maximal regularity property, i.e. for any $\phi \in Y$, the solution of

$$
(0.4) \qquad \begin{cases} v'(t) = A(s)v(t) + \phi(t), & t > s \\ v(s) = 0, \end{cases}
$$

is such that both v' and $A(s)v(\cdot)$ belong to Y. Then a solution of (0.3) may be obtained by the usual fixed point procedure: that is, for any $w \in Y$ such that $A(s)w(\cdot) \in Y$ we set $\Gamma w = v$, where v is the solution of (0.4) with

$$
\phi(t) = [A(t) - A(s)](w(t) + e^{(t-s)A(s)}x),
$$

and we prove that Γ has a fixed point w. Then we may define $G(t, s)x$ for any $x \in X$ by (0.2).

In the above-mentioned papers [DPS] and [LS], the choice $Y = C^\theta([s, s + \delta]; X)$ (with $0 < \theta \leqq \alpha$) lets one define $G(t, s)x$ only for very regular x, i.e. for any x such that $t \to [A(t) - A(s)]e^{(t-s)A(s)}x$ is θ-Hölder continuous up to $t = s$.

Our space Y is described in Section 1; it was introduced in [AT3] for studying the regularity properties of the solutions of $u'(t) = A(t)u(t) + f(t)$, when the operators $A(t)$ have not (necessarily) constant domains. The maximal regularity property in a similar space was also stated in [S2].

Using (0.2), we may easily obtain estimates on $G(t, s)$ and $A(t)G(t, s)$ in several norms, using the well-known estimates on $e^{(t-s)A(s)}$ and the new ones on $W(t, s)$. These estimates let us prove many regularity results for the (strong, classical or strict) solution of the initial value problem

$$
(0.5) \qquad \begin{cases} u'(t) = A(t)u(t) + f(t), & 0 < t \leqq T \\ u(0) = x, \end{cases}
$$

which coincide with the ones of [AT1], [AT2], obtained with completely different methods.

The relatively simple construction of $G(t, s)$ will let us study the asymptotic behavior of $G(t, s)$ as $t \to +\infty$ in the periodic case $A(t) = A(t + T)$ and in the case $\lim_{t \to +\infty} A(t) = A$. This will be the object of a subsequent paper ([L]).

1. Notations and preliminary results

Let X be a real or complex Banach space with norm $\|\cdot\|$ and let $\alpha \in]0, 1[$, $\beta \in [0, 1[$, $a, b \in \mathbf{R}$, $a < b$. We shall consider the usual spaces of functions: $L^\infty(a, b; X)$, $L^1(a, b; X)$, $C([a, b]; X)$, $C(]a, b]; X)$, $C^1([a, b]; X)$, $C^1(]a, b]; V)$, $C^\alpha([a, b]; X)$, $C^{1+\alpha}([a, b]; X)$, $B([a, b]; X) = \{f: [a, b] \to X, \sup_{a \le t \le b} \|f(t)\| < +\infty\}$. Moreover we set:

$$(1.1)\quad Z_{\beta,\alpha}(a, b; X) = \Big\{ f:]a, b] \to X;\ f \in C^\alpha([a + \varepsilon, b]; X)\ \forall \varepsilon \in]0, b - a[, |f|_\beta \doteq \sup_{a < t \le b} (t - a)^\beta \|f(t)\| < +\infty, [f]_{\beta,\alpha} \doteq \sup_{0 < \varepsilon < b - a} \varepsilon^{\alpha+\beta} [f]_{C^\alpha([a+\varepsilon/2, a+\varepsilon]; X)} < +\infty \Big\}$$

where

$$[f]_{C^\alpha([a+\varepsilon/2, a+\varepsilon]; X)} \doteq \sup_{a + \varepsilon/2 \le s < t \le a + \varepsilon} (t - s)^{-\alpha} \|f(s) - f(t)\|.$$

It is not difficult to see that $Z_{\beta,\alpha}(a, b; X)$ is a Banach space with the norm

$$(1.2)\qquad \|f\|_{Z_{\beta,\alpha}(a,b;X)} = |f|_\beta + [f]_{\beta,\alpha}$$

and that there exists $N_1(b - a, \beta, \alpha) > 0$ such that

$$(1.3)\qquad \begin{gathered} \|f\|_{C^\alpha([a+\varepsilon, b]; X)} \le N_1(b - a, \beta, \alpha) \varepsilon^{-(\alpha+\beta)} \|f\|_{Z_{\beta,\alpha}(a,b;X)} \\ \forall f \in Z_{\beta,\alpha}(a, b; X), \quad \forall \varepsilon \in]0, b - a[. \end{gathered}$$

Let $A: D(A) \subset X \to X$ be a linear operator such that†

$$(1.4)\qquad \left\{ \begin{array}{l} \text{there are } \omega \in \mathbf{R},\ \theta \in]\pi/2, \pi],\ M > 0 \text{ such that the resolvent set} \\ \text{of } A \text{ contains the sector } S = \{\lambda \in \mathbf{C}; \lambda \ne \omega, |\arg(\lambda - \omega)| < \theta\} \text{ and} \\ \|(\lambda - \omega)(\lambda - A)^{-1}\|_{L(X)} \le M \text{ for each } \lambda \in S. \end{array} \right.$$

Then A generates an analytic semigroup e^{tA} in X (see [Sin] for analytic semigroups in the case of non-dense domain), and for any $T > 0$ there are $M_0, M_1, M_2 > 0$ (depending on ω, θ, M) such that:

† If X is a real Banach space, condition (1.4) is assumed to hold for the complexification of A.

$$(1.5)\qquad \| t^k A^k e^{tA} x \| \leqq M_k \| x \| \qquad \forall x \in X, \quad t \in [0, T].$$

For $0 < \theta < 1$ we denote by $D_A(\theta, \infty)$, $D_A(\theta)$, $D_A(\theta + 1, \infty)$ the interpolation spaces defined by (see [BB], [Tr, th.1.14.5]):

$$(1.6)\qquad \begin{cases} D_A(\theta, \infty) = \left\{ x \in X;\ [x]_\theta = \sup\limits_{0<t\leq 1} \| t^{1-\theta} A e^{tA} x \| < +\infty \right\}; \\ \| x \|_{D_A(\theta,\infty)} = \| x \| + [x]_\theta; \\ D_A(\theta) = \left\{ x \in D_A(\theta, \infty);\ \lim\limits_{t\to 0} t^{1-\theta} A e^{tA} x = 0 \right\}; \\ \| x \|_{D_A(\theta)} = \| x \|_{D_A(\theta,\infty)}; \\ D_A(\theta + 1, \infty) = \{ x \in D(A);\ Ax \in D_A(\theta, \infty) \}; \\ \| x \|_{D_A(\theta+1,\infty)} = \| x \| + \| Ax \|_{D_A(\theta,\infty)}. \end{cases}$$

Then $D_A(\theta)$ is the closure of $D(A)$ in the norm of $D_A(\theta, \infty)$, for every $\theta \in]0, 1[$. To simplify some notations in the sequel we set $D_A(0, \infty) \doteq X$, $D_A(1,\infty) \doteq D$.

The following interpolation lemma will be used in Section 2.

Lemma 1.1. *Let $A : D(A) \to X$ satisfy* (1.4) *and let $a < b$, $0 < \alpha$, $\theta < 1$, $\alpha \neq \theta$. The for any $u \in C^\alpha([a; b]; D(A)) \cap C^{1+\alpha}([a, b]; X)$ ($D(A)$ is endowed with the graph norm), we have*:

$$u' \in B([a, b]; D_A(\alpha, \infty)), \quad u \in C^{\alpha+1-\theta}([a, b]; D_A(\theta, \infty))$$

and

$$(1.7)\qquad \sup_{a\leq t\leq b} \| u'(t) \|_{D_A(\alpha,\infty)} \leqq N_2(b - a, \alpha)(\| u \|_{C^\alpha([a,b];D(A))} + \| u' \|_{C^\alpha([a,b];X)}),$$

$$(1.8)\qquad \| u \|_{C^{\alpha+1-\theta}([a,b];D_A(\theta,\infty))} \leqq N_3(b - a, \alpha)(\| u \|_{C^\alpha([a,b];D(A))} + \| u' \|_{C^\alpha([a,b];X)}),$$

where

$$N_2(b - a, \alpha) = \max\{1 + 2^\alpha(b - a)^{-\alpha} M_1, \quad M_0 + M_1\},$$

$$N_3(b - a, \alpha) = \max\{N_2(b - a, \alpha) + M_0 + 3,\ 3N_2(b - a, \alpha) + M_0 + M_1\}$$

and M_0, M_1 are given by (1.5) *with $T = 1$. Moreover* (1.8) *holds also for $\alpha = 0$.*

Proof. Let us show (1.7): for any $t \in [a, b]$ we have

$$\sup_{0<\xi\leq 1} \| \xi^{1-\alpha} A e^{\xi A} u'(t) \|$$

$$= \mathrm{Max} \left\{ \sup_{0<\xi<(b-a)/2} \| \xi^{1-\alpha} A e^{\xi A} u'(t) \|, \sup_{(b-a)/2\leq\xi\leq 1} \| \xi^{1-\alpha} A e^{\xi A} u'(t) \| \right\}.$$

If $0<\xi<(b-a)/2$ let h be such that $t+h\in[a,b]$ and $|h|=\xi$. Then:

$$\begin{aligned}
\|\xi^{1-\alpha}Ae^{\xi A}u'(t)\| &\leqq \|\xi^{1-\alpha}Ae^{\xi A}[u'(t)-h^{-1}(u(t+h)-u(t))]\| \\
&\quad + \|\xi^{1-\alpha}Ae^{\xi A}h^{-1}(u(t+h)-u(t))\| \\
&\leqq \left\|\xi^{1-\alpha}Ae^{\xi A}\int_0^1 (u'(t)-u'(t+\sigma h))d\sigma\right\| \\
&\quad + \|\xi^{1-\alpha}Ae^{\xi A}h^{-1}(u(t+h)-u(t))\| \\
&\leqq M_1[u']_{C^\alpha([a,b];X)}+M_0[Au]_{C^\alpha([a,b];X)}.
\end{aligned}$$

If $(b-a)/2\leqq\xi\leqq 1$ we have simply

$$\|\xi^{1-\alpha}Ae^{\xi A}u'(t)\| \leqq \xi^{-\alpha}M_1\|u'(t)\| \leqq 2^\alpha(b-a)^{-\alpha}M_1\sup_{a\leqq t\leqq b}\|u'(t)\|.$$

Now (1.7) follows easily.

Let us show (1.8) for $0\leqq\alpha<\theta$. If $\alpha=0$, for $t, t+h\in[a,b]$, $|h|\leqq 1$ we have

$$\begin{aligned}
&|h|^{\theta-1}\sup_{0<\xi\leqq 1}\|\xi^{1-\theta}Ae^{\xi A}(u(t+h)-u(t))\| \\
&\leqq |h|^{\theta-1}\sup_{0<\xi<|h|}\|\xi^{1-\theta}Ae^{\xi A}(u(t+h)-u(t))\| \\
&\quad + |h|^{\theta-1}\sup_{|h|\leqq\xi\leqq 1}\|\xi^{1-\theta}Ae^{\xi A}(u(t+h)-u(t))\| \\
&\leqq 2M_0\sup_{a\leqq t\leqq b}\|Au(t)\| + M_1\sup_{a\leqq t\leqq b}\|u'(t)\|.
\end{aligned}$$

Analogously, if $0<\alpha<\theta$ and $t, t+h\in[a,b]$, $|h|\leqq 1$, we have:

$$\begin{aligned}
&|h|^{-(\alpha+1-\theta)}\sup_{0<\xi\leqq 1}\|\xi^{1-\theta}Ae^{\xi A}(u(t+h)-u(t))\| \\
&\leqq |h|^{-(\alpha+1-\theta)}\Big[\sup_{0<\xi<|h|}\|\xi^{1-\theta}Ae^{\xi A}(u(t+h)-u(t))\| \\
&\qquad + \sup_{|h|\leqq\xi\leqq 1}\|\xi^{1-\theta}Ae^{\xi A}(u(t+h)-u(t))\|\Big] \\
&\leqq |h|^{-(\alpha+1-\theta)}\Big[\sup_{0<\xi<|h|}\xi^{1-\theta}M_0|h|^\alpha[Au]_{C^\alpha([a,b];X)} + \sup_{|h|\leqq\xi\leqq 1}\xi^{\alpha-\theta}|h| \\
&\qquad \sup_{a\leqq s\leqq b}\|u'(s)\|_{D_A(\alpha,\infty)}\Big] \\
&\leqq M_0[Au]_{C^\alpha([a,b];X)} + N_2(b-a,\alpha)(\|u\|_{C^\alpha([a,b];D(A))} + \|u'\|_{C^\alpha([a,b];X)}).
\end{aligned}$$

Now (1.8) follows easily, recalling that

$$\| u \|_{C^\alpha([a,b];X)} \leqq \max \left\{ \| u \|_{C^1([a,b];X)}, 3 \sup_{a \leqq t \leqq b} \| u(t) \| \right\}$$

and

$$\sup_{0<\xi\leqq 1, a<t<b} \| \xi^{1-\alpha} A e^{\xi A} u(t) \| \leqq M_0 \sup_{a \leqq t \leqq b} \| Au(t) \| .$$

Let now $\theta < \alpha$: we have to show that u' belongs to $C^{\alpha-\theta}([a, b]; D_A(\theta, \infty))$. For $t, t+h \in [a, b]$, $|h| \leqq 1$ we have:

$$\begin{aligned} &|h|^{-(\alpha-\theta)} \sup_{0<\xi\leqq 1} \| \xi^{1-\theta} A e^{\xi A} (u'(t+h) - u(t)) \| \\ &\leqq |h|^{-(\alpha-\theta)} \Bigg[\sup_{0<\xi<|h|} \| \xi^{1-\theta} A e^{\xi A} (u'(t+h) - u'(t)) \| \\ &\qquad + \sup_{|h| \leqq \xi \leqq 1} \| \xi^{1-\theta} A e^{\xi A} (u'(t+h) - u'(t)) \| \Bigg] \\ &\leqq |h|^{-(\alpha-\theta)}. \end{aligned}$$

$$\left[\sup_{0<\xi\leqq |h|} \xi^{\alpha-\theta} \| u'(t+h) - u'(t) \|_{D_A(\alpha,\infty)} + \sup_{|h| \leqq \xi \leqq 1} \xi^{-\theta} M_1 |h|^\alpha [u']_{C^\alpha[(a,b];X)} \right]$$

$$\leqq 2N_2(b-a, \alpha)(\| u \|_{C^\alpha[(a,b];D(A))} + \| u' \|_{C^\alpha([a,b];X)}) + M_1 [u']_{C^\alpha([a,b];X)}$$

and (1.8) follows, recalling that

$$\| u \|_{C^1([a,b]; D_A(\theta,\infty))} = \sup_{a \leqq t \leqq b} \| u(t) \|_{D_A(\theta,\infty)} + \sup_{a \leqq t \leqq b} \| u'(t) \|_{D_A(\alpha,\infty)}. \qquad \square$$

In the following we shall use also some interpolation properties of the spaces $D_A(\beta, \infty)$: namely, if a linear operator B belongs to $L(X) \cap L(D(A))$ then B belongs to $L(D_A(\beta, \infty))$ for any $\beta \in]0, 1[$, and

$$\| B \|_{L(D_A(\beta,\infty))} \leqq N_4(\beta)(\| B \|_{L(X)})^{1-\beta} (\| B \|_{L(D(A))})^\beta . \tag{1.9}$$

Analogously, if B belongs to $L(X, D_A(\beta, \infty)) \cap L(D(A), D_A(\beta, \infty))$ for some $\beta \in [0, 1]$ then B belongs to $L(D_A(\theta, \infty), D_A(\beta, \infty))$ for any $\theta \in]0, 1[$, and

$$\| B \|_{L(D_A(\theta,\infty), D_A(\beta,\infty))} \leqq N_5(\theta, \beta)(\| B \|_{L(X, D_A(\beta,\infty))})^{1-\theta} (\| B \|_{L(D(A), D_A(\beta,\infty))})^\theta . \tag{1.10}$$

Finally, if B belongs to $L(X) \cap L(X, D(A))$ then B belongs also to $L(X, D_A(\beta, \infty))$ for each $\beta \in]0, 1[$, and

$$\|B\|_{L(X,D_A(\beta,\infty))} \leqq N_6(\beta)(\|B\|_{L(X)})^{1-\beta}(\|B\|_{L(X,D(A))})^{\beta}. \tag{1.11}$$

Let now f: $[a, b] \to X$ and let $x \in X$. Consider the initial value problem:

$$\begin{cases} u'(t) = Au(t) + f(t), & a < t \leqq b, \\ u(a) = x. \end{cases} \tag{1.12}$$

It is well known that, under suitable assumptions on f and x, it is possible to prove existence and uniqueness of a solution of (1.12), and also to give maximal regularity results (that is, to show that both u' and $Au(\cdot)$ have the same regularity of f). Here we shall consider a new maximal regularity result, which has been stated in [AT3] for a non-autonomous equation ($u' = A(t)u(t) + f(t)$, $u(0) = x$). We give here a simple direct proof in the time-independent case.

THEOREM 1.2. *Let A satisfy* (1.4) *and let $f \in Z_{\beta,\alpha}(a, b; X)$ with $0 < \beta$, $\alpha < 1$, $x \in D_A(1-\beta, \infty)$. Then the function*

$$u(t) = e^{(t-a)A}x + \int_a^t e^{(t-s)A}f(s)ds, \qquad a \leqq t \leqq b$$

is the unique solution of (1.12), *belongs to $C^{1-\beta}([a, b]; X) \cap B([a, b]; D_A(1-\beta, \infty))$ and it is such that u', $Au(\cdot)$ belong to $Z_{\beta,\alpha}(a, b; X)$. There is $N_7 = N_7(b-a, \beta, \alpha, \omega, \theta, M) > 0$ such that*

$$\|u\|_{C^{1-\beta}([a,b];X)} + \sup_{a \leqq t \leqq b} \|u(t)\|_{D_A(1-\beta,\infty)} + \|u'\|_{Z_{\beta,\alpha}(a,b;X)} + \|Au\|_{Z_{\beta,\alpha}(a,b;X)}$$
$$\leqq N_7(\|x\|_{D_A(1-\beta,\infty)} + \|f\|_{Z_{\beta,\alpha}(a,b;X)}). \tag{1.13}$$

If in addition $\lambda_0 = \sup\{\mathrm{Re}\,\lambda; \lambda \in \sigma(A)\} < 0$, then N_7 can be chosen not depending on $b-a$.

PROOF. Let M_0, M_1, M_2 be given by (1.5) with $T = b - a$, and for $0 < \theta \leqq 1$ let $M_i(\theta)$ ($i = 3, 4, 5, 6$) be such that

$$\begin{cases} \|\xi^{1-\theta}Ae^{\xi A}\|_{L(D_A(\theta,\infty),X)} \leqq M_3(\theta); \quad \|\xi^{2-\theta}A^2e^{\xi A}\|_{L(D_A(\theta,\infty),X)} \leqq M_4(\theta), \\ \qquad\qquad 0 \leqq \xi \leqq b-a; \\ \|e^{\xi A}\|_{L(D_A(\theta,\infty))} \leqq M_5(\theta); \quad \|\xi^{\theta}e^{\xi A}\|_{L(X,D_A(\theta,\infty))} \leqq M_6(\theta), \\ \qquad\qquad 0 \leqq \xi \leqq b-a. \end{cases} \tag{1.14}$$

For $a \leqq t \leqq r \leqq b$ we have

$$\| u(t) \| \leqq M_0 \| x \| + \frac{M_0}{1-\beta}(t-a)^{1-\beta} |f|_\beta; \tag{1.15}$$

and

$$\begin{aligned}
&\| u(t) - u(r) \| \\
&\leqq \left\| \int_{r-a}^{t-a} Ae^{\sigma A} x d\sigma \right\| + \left\| \int_a^r (e^{(t-s)A} - e^{(r-s)A}) f(s) ds \right\| \\
&\quad + \left\| \int_r^t e^{(t-s)A} f(s) ds \right\| \leqq \frac{M_3(1-\beta)}{1-\beta} \| x \|_{D_A(1-\beta,\infty)} (t-r)^{1-\beta} \\
&\quad + \frac{M_1}{1-\beta} \int_0^1 \frac{d\sigma}{(1-\sigma)^{1-\beta}\sigma^\beta} |f|_\beta (t-r)^{1-\beta} + \frac{M_0}{1-\beta} |f|_\beta (t-r)^{1-\beta}
\end{aligned}$$

so that u belongs to $C^{1-\beta}([a,b];X)$. For $a \leqq t \leqq b$ we have

$$\| u(t) \|_{D_A(1-\beta,\infty)} \leqq M_5(1-\beta) \| x \|_{D_A(1-\beta,\infty)} + M_6(1-\beta) \int_0^1 \frac{d\sigma}{(1-\sigma)^{1-\beta}\sigma^\beta} |f|_\beta$$

so that u is bounded with values in $D_A(1-\beta,\infty)$.

Using (1.5) it is easy to see that $u(t) \in D(A)$ and that u is differentiable with values in X for $t > a$. Moreover, for $a \leqq t \leqq b$ we have:

$$\begin{aligned}
&(t-a)^\beta \| Au(t) \| \\
&\leqq M_3(1-\beta) \| x \|_{D_A(1-\beta,\infty)} + (t-a)^\beta \left\| \int_a^{(a+t)/2} Ae^{(t-s)A}(f(s) - f(t)) ds \right\| \\
&\quad + (t-a)^\beta \left\| \int_{(a+t)/2}^t Ae^{(t-s)A}(f(s) - f(t)) ds \right\| \\
&\quad + (t-a)^\beta (e^{(t-a)A} - 1) f(t) \| \\
&\leqq M_3(1-\beta) \| x \|_{D_A(1-\beta,\infty)} + M_1 \int_a^{(a+t)/2} \left[\left(\frac{t-a}{s-a} \right)^\beta + 1 \right] \frac{ds}{t-s} |f|_\beta \\
&\quad + M_1 \int_{(a+t)/2}^t \frac{ds}{(t-a)^\alpha (t-s)^{1-\alpha}} [f]_{\beta,\alpha} + (M_0 + 1) |f|_\beta \\
&\leqq M_3(1-\beta) \| x \|_{D_A(1-\beta,\infty)} + \left[M_1 \left(\int_0^{1/2} \frac{d\sigma}{(1+\sigma)\sigma^\beta} + \log 2 \right) + M_0 + 1 \right] |f|_\beta \\
&\quad + \frac{2^\alpha}{\alpha} M_1 [f]_{\beta,\alpha}.
\end{aligned}$$

Moreover, for $a < a + \varepsilon/2 \leqq r \leqq t \leqq a + \varepsilon \leqq b$ we have

$$
\begin{aligned}
&\varepsilon^{\alpha+\beta} \| Au(t) - Au(r) \| \\
&\leqq \varepsilon^{\alpha+\beta} \left\| \int_{r-a}^{t-a} A^2 e^{\sigma A} x d\sigma \right\| \\
&\quad + \varepsilon^{\alpha+\beta} \left\| \int_a^{(a+r)/2} (Ae^{(t-s)A} - Ae^{(r-s)A})(f(s) - f(r)) ds \right\| \\
&\quad + \varepsilon^{\alpha+\beta} \left\| \int_{(a+r)/2}^{r} (Ae^{(t-s)A} - Ae^{(r-s)A})(f(s) - f(r)) ds \right\| \\
&\quad + \varepsilon^{\alpha+\beta} \left\| A \int_a^r e^{(t-s)A} (f(r) - f(t)) ds \right\| \\
&\quad + \varepsilon^{\alpha+\beta} \left\| \int_r^t Ae^{(t-s)A} (f(s) - f(t)) ds \right\| \\
&\quad + \varepsilon^{\alpha+\beta} \| (e^{(r-a)A} - 1)(f(r) - f(t)) \| \\
&\quad + \varepsilon^{\alpha+\beta} \left\| \int_{r-a}^{t-a} Ae^{\sigma A} f(t) d\sigma \right\| \\
&\leqq 2^{\alpha+\beta} M_4(1-\beta) \int_{r-a}^{t-a} \frac{d\sigma}{\sigma^{1-\alpha}} \| x \|_{D_A(1-\beta,\infty)} \\
&\quad + 2^{2\alpha+\beta} M_2 \int_a^{(a+r)/2} \left\{ \left[\left(\frac{r-a}{s-a} \right)^{\beta} + 1 \right] \int_{r-s}^{t-s} \frac{d\sigma}{\sigma^{2-\alpha}} \right\} ds \, |f|_{\beta} \\
&\quad + M_2 \cdot 2^{\alpha+\beta} \int_{(a+r)/2}^{r} (r-s)^{\alpha} \int_{r-s}^{t-s} \frac{d\sigma}{\sigma^2} ds [f]_{\beta,\alpha} \\
&\quad + M_1 \int_r^t \frac{ds}{(t-s)^{1-\alpha}} [f]_{\beta,\alpha} \\
&\quad + 2(M_0 + 1)[f]_{\beta,\alpha}(t-r)^{\alpha} + 2^{\alpha+\beta} M_1 \int_{r-a}^{t-a} \frac{d\sigma}{\sigma^{1-\alpha}} |f|_{\beta} \\
&\leqq \left\{ \frac{2^{\alpha+\beta}}{\alpha} M_4(1-\beta) \| x \|_{D_A(1-\beta,\infty)} \right. \\
&\quad + \left[\frac{2^{2\alpha+\beta}}{\alpha} M_2 \left(\int_0^{1/2} \frac{d\sigma}{(1-\sigma)\sigma^{\beta}} + \log 2 \right) + \frac{2^{\alpha+\beta}}{\alpha} M_1 \right] |f|_{\beta} \\
&\quad \left. + \left[2^{\alpha+\beta} M_2 \int_0^{+\infty} \frac{d\tau}{\tau^{1-\alpha}(1+\tau)} + \frac{M_1}{\alpha} + 2(M_0+1) \right] [f]_{\beta,\alpha} \right\} (t-r)^{\alpha}
\end{aligned}
$$

so that $Au(\cdot)$ belongs to $Z_{\beta,\alpha}(a,b;X)$ and (1.13) holds. N_7 depends on $b-a$ through the constants M_i ($i=0,1,\ldots,6$) and the term

$$\frac{M_0}{1-\beta}(b-a)^{1-\beta}$$

which comes from estimating the sup norm of u (see (1.15)). If $\lambda_0<0$, the M_i's can be chosen independent on $b-a$. In particular, M_0 can be chosen such that

$$\| e^{tA} \|_{L(X)} \leqq M_0 e^{\lambda_0 t/2} \qquad \forall t \geqq 0.$$

Therefore estimate (1.15) may be replaced by

$$\| u(t) \| \leqq M_0 \| x \| + M_0[1/(1-\beta)+2/|\lambda_0|]|f|_\beta$$

so that N_7 is independent on $b-a$. □

Let now D be a continuously embedded subspace of X and $t_0<t_1$. We shall consider a family of linear operators $A(t): D\to X$ ($t_0\leqq t\leqq t_1$) such that:

$$(1.16)\qquad \begin{cases} \text{for each } t\in[t_0,t_1],\ A(t):D\to X \text{ satisfies (1.4);} \\ \text{the graph norm of } A(t) \text{ is equivalent to the norm of } D, \end{cases}$$

$$(1.17)\qquad \text{the function } t\to A(t) \text{ belongs to } C^\alpha([t_0,t_1];L(D,X)).$$

In the sequel we shall often write for brevity:

$$(1.18)\qquad \|A\|_\infty=\sup_{t_0\leqq t\leqq t_1}\|A(t)\|_{L(D,X)},\quad [A]_\alpha=[A]_{C^\alpha([t_0,t_1];L(D,X))}.$$

Assumptions (1.16) and (1.17) imply easily that

$$(1.19)\qquad \begin{cases} \text{there are } \bar\omega\in\mathbf{R},\ \bar\theta\in]\pi/2,\pi],\ \bar M>0 \text{ such that, for any } t\in[t_0,t_1], \\ A(t) \text{ satisfies (1.4) with constants } \omega=\bar\omega,\ \theta=\bar\theta,\ M=\bar M. \end{cases}$$

In its turn, (1.19) implies that $D_{A(t)}(\beta,\infty)=D_{A(t_0)}(\beta,\infty)$ for any $\beta\in]0,1[$, and there are $\nu\geqq 1$, $\nu(\beta)\geqq 1$ such that

$$(1.20)\qquad \begin{cases} \text{(i)}\ \dfrac{1}{\nu(\beta)}\|x\|_{D_{A(t_0)}(\beta,\infty)}\leqq \|x\|_{D_{A(t)}(\beta,\infty)}\leqq \nu(\beta)\|x\|_{D_{A(t_0)}(\beta,\infty)}, \\ \qquad 0<\beta<1,\quad x\in D_{A(t_0)}(\beta,\infty),\quad t_0\leqq t\leqq t_1; \\ \text{(ii)}\ \dfrac{1}{\nu}\|x\|_D\leqq \|x\| + \|A(t)x\| \leqq \nu\|x\|_D,\ x\in D,\ t_0\leqq t\leqq t_1. \end{cases}$$

It can also been shown that the following estimates hold for $0 \leqq s \leqq t_1 - t_0$, $t_0 \leqq r < t \leqq t_1$:

$$
(1.21) \quad \begin{cases}
\text{(i)} \ \| s^k (A(t))^k e^{sA(t)} \|_{L(X)} \leqq M_k, \quad k = 0, 1, 2; \\
\text{(ii)} \ \| s^{\max\{0, \beta - \theta\}} e^{sA(t)} \|_{L(D_{A(t_0)}(\theta, \infty), D_{A(t_0)}(\beta, \infty))} \leqq M_3(\theta, \beta), \\
\qquad k = 1, 2; \quad 0 \leqq \theta, \beta \leqq 1; \\
\text{(iii)} \ \| s^{k+\beta-\theta} (A(t))^k e^{sA(t)} \|_{L(D_{A(t_0)}(\theta, \infty), D_{A(t_0)}(\beta, \infty))} \leqq M_4(\theta, \beta), \\
\qquad k = 1, 2; \quad 0 \leqq \theta, \beta \leqq 1; \\
\text{(iv)} \ \| s^{\max\{0, \beta - \theta\}} (e^{sA(t)} - e^{sA(r)}) \|_{L(D_{A(t_0)}(\theta, \infty), D_{A(t_0)}(\beta, \infty))} \\
\qquad \leqq M_5(\theta, \beta)(t - r)^\alpha, \quad 0 \leqq \theta, \beta \leqq 1; \\
\text{(v)} \ \| s^{k+\beta-\theta} ((A(t))^k e^{sA(t)} - (A(r))^k e^{sA(r)}) \|_{L(D_{A(t_0)}(\theta, \infty), D_{A(t_0)}(\beta, \infty))} \\
\qquad \leqq M_6(\theta, \beta)(t - r)^\alpha, \quad 0 \leqq \theta, \beta \leqq 1, \quad k = 1, 2.
\end{cases}
$$

See [Sin] for the proof of (i), (ii), (iii) and [AT2] for the proof of (iv), (v).

Let $f \in C([t_0, t_1]; X)$ and $x \in X$. We shall study the problem

$$
(1.22) \quad \begin{cases} u'(t) = A(t)u(t) + f(t), \quad t_0 < t \leqq t_1, \\ u(t_0) = x. \end{cases}
$$

A function $u \in C([t_0, t_1]; X)$ is said to be a *strict* (resp. *classical*) solution of (1.22) if u belongs also to $C([t_0, t_1]; D) \cap C^1([t_0, t_1]; X)$ (resp. to $C(]t_0, t_1]; D) \cap C^1(]t_0, t_1]; X)$) and satisfies (1.22). u is said to be a *strong* solution of (1.22) if $u(t_0) = x$ and there are $u_n \in C([t_0, t_1]; D) \cap C^1([t_0, t_1]; X)$, $n \in \mathbf{N}$, such that

$$
\lim_{n \to \infty} u_n = u \quad \text{and} \quad \lim_{n \to \infty} u_n' - A(\cdot)u_n(\cdot) = f \quad \text{in } C([t_0, t_1]; X).
$$

The same definition of classical solution may be given when $f \in C(]t_0, t_1]; X)$.

It is easy to see that a necessary condition for the existence of a strict solution of (1.22) is $x \in D$, $A(t_0)x + f(t_0) \in \bar{D}$ and a necessary condition for the existence of a classical or strong solution of (1.22) is $x \in \bar{D}$.

It will be seen later that, when f is continuous in $[t_0, t_1]$, then any classical solution is a strong one. Also, the uniqueness of the strong solution of (1.22) will be shown. By now we limit ourselves to state a uniqueness lemma which will be used in the sequel.

LEMMA 1.3. *Let u be a classical solution of*

$$
\begin{cases} u'(t) = A(t)u(t), \quad t_0 < t \leqq t_1 \\ u(t_0) = 0 \end{cases}
$$

and assume that

$$\{u\}_\beta = \sup_{t_0 \leqq \bar{t} < t \leqq t_1} (t-\bar{t})^\beta \, \| u(t) \|_D < +\infty \quad \textit{for some } \beta \in]0,1[.$$

Then $u(t) = 0 \ \forall t \in [t_0, t_1]$.

PROOF. Assume by contradiction that $\bar{t} = \sup\{t \geqq t_0,\ u(s) = 0$ for $t_0 \leqq s \leqq t\} < t_1$. Then since

$$\frac{d}{ds} e^{(t-s)A(t)} u(s) = e^{(t-s)A(t)}(A(s) - A(t))u(s) \quad \text{for } \bar{t} < s < t,$$

we have:

$$A(t)u(t) = A(t)e^{(t-\bar{t})A(t)}u(\bar{t}) + \int_{\bar{t}}^{t} A(t)e^{(t-s)A(t)}(A(s) - A(t))u(s)ds, \quad \bar{t} < t \leqq t_1,$$

so that

$$\begin{aligned}(t-\bar{t})^\beta \, \| A(t)u(t) \| &\leqq M_1[A]_\alpha (t-\bar{t})^\beta \int_{\bar{t}}^{t} \frac{ds}{(t-s)^{1-\alpha}(s-\bar{t})^\beta} \{u\}_\beta \\ &= M_1[A]_\alpha \int_0^1 \frac{d\sigma}{(1-\sigma)^{1-\alpha}\sigma^\beta} \{u\}_\beta (t-\bar{t})^\alpha.\end{aligned}$$

and

$$\begin{aligned}(t-E)^\beta \, \| u(t) \| &\leqq M_0[A]_\alpha (t-\bar{t})^\beta \int_{\bar{t}}^{t} \frac{(t-s)^\alpha}{(s-\bar{t})^\beta} ds \{u\}_\beta \\ &= M_0[A]_\alpha \int_0^1 (1-\sigma)^\alpha \sigma^{-\beta} d\sigma \{u\}_\beta (t-\bar{t})^{1+\alpha}\end{aligned}$$

Therefore u vanishes in $[\bar{t}, (\bar{t}+\delta) \wedge t_1]$, where

$$\delta = \left(\nu \left(M_1[A]_\alpha \int_0^1 \frac{d\sigma}{(1-\sigma)^{1-\alpha}\sigma^\beta} + M_0[A]_\alpha \int_0^1 (1-\sigma)^\alpha \sigma^{-\beta} d\sigma (t_1 - t_0) \right) \right)^{-1/\alpha}.$$

This contradicts the definition of $\bar{t}$. Hence $\bar{t} = t_1$. □

2. The evolution operator *G(t, s)*

Through this section (1.16) and (1.17) will be assumed, and estimates (1.20), (1.21) will be used. We begin recalling a result which has been proved in [LS].

PROPOSITION 2.1. *Let* $t_0 \leqq s < t_1$ *and let* $\phi \in C^\theta([s, t_1]; X)$, $y \in D$, $A(s)y + \phi(s) \in D_{A(0)}(\theta, \infty)$, with $0 < \theta \leqq \alpha$. *Then problem*

$$(2.1) \qquad \begin{cases} u'(t) = A(t)u(t) + \phi(t), & s \leqq t \leqq t_1 \\ u(s) = y \end{cases}$$

has a unique strict solution $u \in C^\theta([s,t_1];D) \cap C^{1+\theta}([s,t_1];X)$, *and there is* $N_8(\theta)$ (depending also $t_1 - t_0$, ν, α, $\|A\|_{C^\alpha([t_0,t_1];L(D,X))}$, M_i $(i=0,1,2)$) *such that*

$$\|u\|_{C^\theta([s,t_1];D)} + \|u'\|_{C^\theta([s,t_1];X)}$$
$$\leqq N_8(\theta)(\|\phi\|_{C^\theta([s,t_1];X)} + \|y\|_D + \|A(s)y + \phi(s)\|_{D_{A(s)}(\theta,\infty)}). \tag{2.2}$$ □

With the aid of Proposition 2.1 (taking $\phi = 0$) we could define $G(t,s)y$ only for very regular y (i.e., for $y \in D_{A(s)}(\theta+1,\infty)$). To extend $G(t,s)$ over the whole space X, let us consider problem $u'(t) = A(t)u(t) + f(t)$ $(s < t \leqq t_1)$, $u(s) = x$: if u is a classical solution, then the function $w(t) = u(t) - e^{(t-s)A(s)}x$ satisfies

$$\begin{cases} w'(t) = A(t)w(t) + (A(t)-A(s))e^{(t-s)A(s)}x + f(t), \quad s < t \leqq t_1, \\ w(s) = 0. \end{cases} \tag{2.3}$$

Now we study problem (2.3) using the maximal regularity result of Theorem 1.2.

PROPOSITION 2.2. *Let* $x \in D_{A(t_0)}(\eta,\infty)$ $(0 \leqq \eta < 1-\alpha)$, *and let* $f \in Z_{\beta,\alpha}(s,t_1;X)$, *with* $\beta = 1-\alpha-\eta$. *Then problem* (2.3) *has a classical solution* $w \in C^{1-\beta}([s,t_1];X) \cap B([s,t_1];D_{A(t_0)}(\alpha+\eta,\infty)) \cap Z_{\beta,\alpha}(s,t_1;D)$, *such that* w' *belongs to* $Z_{\beta,\alpha}(s,t_1;X)$. *There is* $N_9(\eta)$ *(depending also on* $t_1 - t_0$, $\|A\|_{C^\alpha}$, $\bar{M}$, $\bar{\theta}$, $\bar{\omega}$, ν, α), *such that:*

$$\|w\|_{C^{1-\beta}([s,t_1];X)} + \sup_{s \leqq t \leqq t_1} \|w(t)\|_{D_{A(t_0)}(\alpha+\eta,\infty)} + \|w\|_{Z_{\beta,\alpha}(s,t_1;D)} + \|w'\|_{Z_{\beta,\alpha}(s,t_1;X)}$$
$$\leqq N_9(\eta)(\|x\|_{D_{A(t_0)}(\eta,\infty)} + \|f\|_{Z_{\beta,\alpha}(s,t_1;X)}). \tag{2.4}$$

Moreover w *is the unique solution of* (2.3) *in the class of all* $v \in C(]s,t_1];D)$ *such that* $\|(t-s)^\beta v(t)\|_D$ *is bounded.*

PROOF. First we solve (2.3) in an interval $[s, s+\delta]$ (with δ small) by means of a fixed point theorem in the space $Z_{\beta,\alpha}(s,s+\delta;D)$. For any $\delta \in]0, t_1 - s[$ and $v \in Z_{\beta,\alpha}(s,s+\delta;D)$ the function $t \to (A(t)-A(s))v(t)$ belongs to $Z_{\beta,\alpha}(s,s+\delta;X)$. Actually, for $s < s+\varepsilon/2 \leqq r < t \leqq s+\varepsilon \leqq t_1$ we have:

$$\begin{cases} \text{(i) } (t-s)^\beta \|(A(t)-A(s))v(t)\| \leqq [A]_\alpha \delta^\alpha |v|_\beta, \\ \text{(ii) } \varepsilon^{\alpha+\beta} \|(A(t)-A(s))v(t) - (A(r)-A(s))v(r)\| \\ \quad \leqq \varepsilon^{\alpha+\beta} \|(A(t)-A(r))v(r)\| + \varepsilon^{\alpha+\beta} \|(A(t)-A(s))(v(t)-v(r))\| \\ \quad \leqq 2^\beta [A]_\alpha (t-r)^\alpha \delta^\alpha |v|_\beta + [A]_\alpha \delta^\alpha [v]_{\beta,\alpha} (t-r)^\alpha. \end{cases} \tag{2.5}$$

Also the function $t \to (A(t)-A(s))e^{(t-s)A(s)}x$ belongs to $Z_{\beta,\alpha}(s, s+\delta; X)$ for any $x \in D_{A(t_0)}(\beta, \infty)$ and we have, for $s < s+\varepsilon/2 \leqq r \leqq t \leqq s+\varepsilon \leqq t_1$:

$$
(2.6)\quad \begin{cases} \text{(i)}\ (t-s)^\beta \| (A(t)-A(s))e^{(t-s)A(s)}x \| \leqq M_3(\eta, 1)[A]_\alpha \| x \|_{D_{A(t_0)}(\eta,\infty)}, \\ \text{(ii)}\ \varepsilon^{\alpha+\beta} \| (A(t)-A(s))e^{(t-s)A(s)}x - (A(r)-A(s))e^{(r-s)A(s)}x \| \\ \quad \leqq \varepsilon^{\alpha+\beta} \| (A(t)-A(r))e^{(r-s)A(s)}x \| \\ \qquad + \varepsilon^{\alpha+\beta} \| (A(r)-A(s))(e^{(t-s)A(s)} - e^{(r-s)A(s)})x \| \\ \quad \leqq 2^{1-\eta} M_3(\eta, 1)[A]_\alpha (t-r)^\alpha \| x \|_{D_{A(t_0)}(\eta,\infty)} \\ \qquad + \alpha^{-1} 2^{1-\eta} M_4(\eta, 1)[A]_\alpha (t-r)^\alpha \| x \|_{D_{A(t_0)}(\eta,\infty)}. \end{cases}
$$

Finding a solution of (2.3) in the space $Z_{\beta,\alpha}(s, s+\delta; D)$ is equivalent to finding a fixed point of the operator $\Gamma : Z_{\beta,\alpha}(s, s+\delta, D) \to Z_{\beta,\alpha}(s, s+\delta; D)$, $\Gamma v = u$, where u is the solution of

$$
\begin{cases} u'(t) = A(s)u(t) + [A(t)-A(s)]v(t) + [A(t)-A(s)]e^{(t-s)A(s)}x + f(t), \\ \qquad\qquad s < t \leqq s+\delta, \\ u(s) = 0. \end{cases}
$$

By (2.5), (1.13), and (1.20) (ii) we have

$$
\| \Gamma \|_{L(Z_{\beta,\alpha}(s,s+\delta;D))} \leqq \nu N_7(t_1 - t_0, \beta, \alpha, \bar{\omega}, \bar{\theta}, \bar{M})(2^\beta + 1)[A]_\alpha \delta^\alpha,
$$

so that Γ is a contraction with constant $\frac{1}{2}$ if

$$
\delta \leqq (2\nu N_7(t_1 - t_0, \beta, \alpha, \bar{\omega}, \bar{\theta}, \bar{M})(2^\beta + 1)[A]_\alpha)^{-1/\alpha}.
$$

In this case Γ has a unique fixed point v, which belongs also to $C^{1-\beta}([s, s+\delta]; X) \cap B([s, s+\delta]; D_{A(t_0)}(1-\beta, \infty))$ by Theorem 1.2. Thanks to (2.6), (1.13) and (1.20), we have

$$
\begin{aligned}
&\| v \|_{C^{1-\beta}([s,s+\delta];X)} + \sup_{s \leqq t \leqq s+\delta} \| v(t) \|_{D_{A(t_0)}(1-\beta,\infty)} + \| v' \|_{Z_{\beta,\alpha}(s,s+\delta;X)} \\
&\quad + \| A(s)v \|_{Z_{\beta,\alpha}(s,s+\delta,X)} \\
(2.7)\qquad &\leqq 2N_7(t_1 - t_0, \beta, \alpha, \bar{\omega}, \bar{\theta}, \bar{M})[(3M_3(\eta, 1) \\
&\qquad + 2^{1-\eta}\alpha^{-1} M_4(\eta, 1)[A]_\alpha) \| x \|_{D_{A(t_0)}(\eta,\infty)} + \| f \|_{Z_{\beta,\alpha}(s,s+\delta;X)}] \\
&= N_{10} \| x \|_{D_{A(t_0)}(\eta,\infty)} + N_{11} \| f \|_{Z_{\beta,\alpha}(s,s+\delta;X)}.
\end{aligned}
$$

If $s+\delta = t_1$ the proof is finished. Otherwise, set $s_1 = s+\delta$, $y = w(s+\delta)$, $\phi(t) = (A(t)-A(s))e^{(t-s)A(s)}x + f(t)$. Then y belongs to D, and $A(s_1)y +$

$\phi(s_1) = v'(s_1)$ belongs to $D_{A(0)}(\alpha, \infty)$ thanks to Lemma 1.1 (since v belongs to $C^{\alpha}([s+\varepsilon, s_1]; D) \cap C^{1+\alpha}([s+\varepsilon, s_1]; X)$ for any $\varepsilon \in]0, s_1 - s[$). Using (2.7), (1.3) and (1.7) with $a = s + \delta/2$, $b = s + \delta = s_1$ we get

$$\| A(s_1)y + \phi(s_1) \|_{D_{A(t_0)}(\alpha,\infty)} \leqq 2N_2(t_1 - t_0, \alpha)N_1(t_1 - t_0, \beta, \alpha))$$
$$\times \delta^{-1}(N_{10} \| x \| \; \| x \|_{D_{A(t_0)}}(\eta, \infty) + N_{11} \| f \|_{Z_{\beta,\alpha}(t_0,t_1;X)}).$$

Moreover ϕ belongs to $C^{\alpha}([s_1, t_1]; X)$, and by (2.6) and (1.3) we get

$$\| \phi \|_{C^{\alpha}([s_1,t_1];X)}$$
$$\leqq N_1(t_1 - t_0, \beta, \alpha)\delta^{-1}[3M_3(\eta, 1) + 2^{1-\eta}\alpha^{-1}M_4(\eta, 1))[A]_{\alpha}] \| x \|_{D_{A(t_0)}(\eta,\infty)}$$
$$+ \| f \|_{Z_{\beta,\alpha}(s,t_1;X)}.$$

By Proposition 2.1, problem (2.1) has a unique strict solution

$$u \in C^{\alpha}([s_1, t_1]; D) \cap C^{1+\alpha}([s_1, t_1]; X),$$

and

$$\| u \|_{C^{\alpha}([s_1,t_1];D)} + \| u' \|_{C^{\alpha}([s_1,t_1];X)}$$
$$\leqq N_8(\alpha)(\| \phi \|_{C^{\alpha}([s_1,t_1];X)} + \| y \|_D + \| A(s_1)y + \phi(s_1) \|_{D_{A(t_0)}(\alpha,\infty)})$$
$$\leqq N_8[1 + 2N_2N_1\delta^{-1} + N_1/(2\delta N_7)]$$
$$\times (N_{10} \| x \| \; \| x \|_{D_{A(t_0)}(\eta,\infty)} + N_{11} \| f \|_{Z_{\beta,\alpha}(t_0,t_1;X)}).$$

Setting now

$$w(t) = \begin{cases} v(t), & s \leqq t \leqq s + \delta \\ u(t), & s + \delta \leqq t \leqq t_1 \end{cases}$$

the conclusion follows easily. □

Corollary 2.3. *Let $0 < \theta < 1$. Then*

(i) *for any $f \in Z_{1-\theta,\theta}(t_0, t_1; X)$ and $x \in \bar{D}$, problem* (1.22) *has a classical solution,*

(ii) *for any $f \in C^{\theta}([t_0, t_1]; X)$ and $x \in D$ such that $A(t_0)x + f(t_0) \in \bar{D}$, problem* (1.22) *has a strict solution, which belongs also to $C^{\alpha\wedge\theta}([t_0 + \varepsilon, t_1]; D) \cap C^{1+(\alpha\wedge\theta)}([t_0 + \varepsilon, t_1]; X)$ for any $\varepsilon \in]0, t_1 - t_0[$;*

(iii) *for any $f \in C^{\theta}([t_0, t_1]); X)$ and $x \in D$ such that $A(t_0)x + f(t_0) \in D_{A(0)}(\theta, \infty)$, problem* (1.22) *has a strict solution belonging to $C^{\alpha\wedge\beta}([t_0, t_1]; D) \cap C^{1+(\alpha\wedge\theta)}([t_0, t_1]; X)$.*

PROOF. (i) It is sufficent to set $u(t) = w(t) + e^{(t-t_0)A(t_0)}x$, where w is the solution of (2.3), with $s = t_0$, given by Proposition 2.2.

(ii) Let $u(t) = v(t) + z(t)$, where $v(t) = e^{(t-t_0)A(t_0)}x + \int_{t_0}^{t} e^{(t-s)A(t_0)}f(t_0)ds$ and $z(t)$ are respectively the solutions of

$$\begin{cases} v'(t) = A(t_0)v(t) + f(t_0), & t_0 \leqq t \leqq t_1, \\ v(t_0) = x; \end{cases}$$

$$\begin{cases} z'(t) = A(t)z(t) + (A(t) - A(t_0))v(t) + f(t) - f(t_0), & t_0 \leqq t \leqq t_1, \\ z(t_0) = 0. \end{cases}$$

Then v belongs to $C([t_0, t_1]; D) \cap C^1([t_0, t_1]; X) \cap C^\infty(]t_0, t_1]; D)$, and, using estimates (1.21), it is easy to see that $(A(\cdot) - A(t_0))v(\cdot)$ belongs to $C^{\alpha \wedge \theta}([t_0, t_1]; X)$. Proposition 2.1 may be applied now, to find $z \in C^{\alpha \wedge \theta}([t_0, t_1]; D) \cap C^{1+(\alpha \wedge \theta)}([t_0, t_1]; X)$.

Therefore $u(t)$ is a strict solution of (1.22) with the regularity properties claimed.

To get (iii) it is sufficient to apply Proposition 2.1 with $s = t_0$, $y = x$ and $\phi = f$. □

DEFINITION 2.4. Let $t_0 \leqq s \leqq t \leqq t_1$ and $x \in X$. Set

$$W(t, s)x = w(t) \tag{2.8}$$

where w is the solution of (2.3) (with $f = 0$) given by Proposition 2.2.

Set also:

$$G(t, s)x = W(t, s)x + e^{(t-s)A(s)}x. \tag{2.9}$$

Here we list some estimates on $G(t, s)x$ and its partial derivative

$$\frac{\partial}{\partial t} G(t, s)x = A(t)G(t, s)x \qquad (t > s),$$

which are similar to the well-known ones in the time-independent case.

PROPOSITION 2.5. *There are constants (depending also on α, $t_1 - t_0$, ν, $\| A \|_{C^\nu([t_0,t_1];L(D,X))}$, M_k $(k = 0, 1, \ldots, 4)$) such that for $t_0 \leqq s < r < t \leqq t_1$ we have:*

$$\| G(t, s) \|_{L(D_{A(t_0)}(\theta,\infty), D_{A(t_0)}(\beta,\infty))} \leqq \frac{c_0(\theta, \beta)}{(t-s)^{\beta-\theta}}, \quad 0 \leqq \theta \leqq \beta \leqq 1; \tag{2.10}$$

$$\| A(t)G(t, s) \|_{L(D_{A(t_0)}(\theta,\infty), D_{A(t_0)}(\beta,\infty))} \leqq \frac{c_1(\theta, \beta)}{(t-s)^{1+\beta-\theta}}, \ 0 \leqq \theta \leqq 1, 0 \leqq \beta \leqq \alpha; \tag{2.11}$$

$$\tag{2.12} \begin{aligned} &\| G(t,s) - G(r,s) \|_{L(D_{A(t_0)}(\theta,\infty), D_{A(t_0)}(\beta,\infty))} \\ &\leqq c_2(\theta,\beta)\left[\frac{(t-r)^{1-\beta+\theta(\alpha\wedge\beta)}}{(r-s)^{(1-\alpha)(1-\theta)}} + \int_{r-s}^{t-s} \frac{d\sigma}{\sigma^{\beta-\theta+1}}\right], \quad 0 \leqq \theta, \beta \leqq 1; \end{aligned}$$

$$\tag{2.13} \begin{aligned} &\| A(t)G(t,s) - A(r)G(r,s) \|_{L(D_{A(t_0)}(\theta,\infty), D_{A(t_0)}(\beta,\infty))} \\ &\leqq c_3(\theta,\beta)\left[\frac{(t-r)^{\alpha-\beta}}{(r-s)^{1-\theta}} + \int_{r-s}^{t-s} \frac{d\sigma}{\sigma^{2+\beta-\theta}}\right], \quad 0 \leqq \theta \leqq 1, \ 0 \leqq \beta \leqq \alpha; \end{aligned}$$

$$\tag{2.14} \| A(t)G(t,s) \|_{L(D_{A(s)}(\theta+1,\infty), D_{A(t_0)}(\beta,\infty))} \leqq \frac{c_4(\theta,\beta)}{(t-s)^{\max\{0,\beta-\theta\}}}, \quad 0 \leqq \theta \leqq 1, \quad 0 \leqq \beta \leqq \alpha;$$

$$\tag{2.15} \begin{aligned} &\| A(t)G(t,s) - A(r)G(r,s) \|_{L(D_{A(s)}(\theta+1,\infty), D_{A(t_0)}(\beta,\infty))} \\ &\leqq c_5(\theta,\beta)\left[(t-r)^{\alpha-\beta} + \int_{r-s}^{t-s} \frac{d\sigma}{\sigma^{1+\beta-\theta}}\right], \quad 0 \leqq \theta \leqq 1, \ 0 \leqq \beta \leqq \alpha. \end{aligned}$$

PROOF. We use (2.9) and the equality

$$\tag{2.16} A(t)G(t,s)x = \frac{\partial}{\partial t} W(t,s)x + A(s)e^{(t-s)A(s)}, \qquad x \in X, \quad t > s$$

and we study separately the estimates for $W(t,s)$ and $e^{(t-s)A(s)}x$. By (2.2) it follows for any $x \in X$, using respectively (1.11), (1.8), (1.7)–(1.3), (1.8) again:

$$\tag{2.17} \begin{cases} \text{(i) } \| W(t,s)x \|_{D_{A(t_0)}(\beta,\infty)} \leqq N_6(\beta)N_7(\alpha)(t-s)^{-\beta(1-\alpha)}N_9(0)\|x\|, \\ \qquad t_0 \leqq s < t \leqq t_1, \quad 0 \leqq \beta \leqq 1; \\ \text{(ii) } \| W(t,s)x - W(r,s)x \|_{D_{A(t_0)}(\beta,\infty)} \\ \qquad \leqq N_3(t_1-t_0,0)N_7(\alpha)N_9(0)(r-s)^{\alpha-1}(t-r)^{1-\beta}\|x\|, \\ \qquad t_0 \leqq s < r < t \leqq t_1, \quad 0 \leqq \beta \leqq 1; \\ \text{(iii) } \left\| \frac{\partial}{\partial t} W(t,s)x \right\|_{D_{A(t_0)}(\beta,\infty)} \\ \qquad \leqq N_2(t_1-t_0,\beta)N_1(t_1-t_0,1-\beta,\beta)N_9(0)N_7(\beta)(t-s)^{-1}\|x\|, \\ \qquad t_0 \leqq s < t \leqq t_1, \quad 0 \leqq \beta \leqq \alpha; \\ \text{(iv) } \left\| \frac{\partial}{\partial t} W(t,s)x \,\Big|_{t=r_2} - \frac{\partial}{\partial t} W(t,s)x \,\Big|_{t=r_1} \right\|_{D_{A(t_0)}(\beta,\infty)} \\ \qquad \leqq N_9(0)N_3(t_1-t_0,\alpha)N_1(t_1-t_0,1-\alpha,\alpha) \\ \qquad \times N_7(\alpha)(r_1-s)^{-1}(r_2-r_1)^{\alpha-\beta}\|x\|, \\ \qquad t_0 \leqq s < r_1 \leqq r_2 \leqq t_1, \quad 0 \leqq \beta \leqq \alpha. \end{cases}$$

For any $x \in D$ and $s \in [t_0, t_1]$ the function

$$\phi(t) = (A(t) - A(s))e^{(t-s)A(s)}x \qquad (s \leqq t \leqq t_1)$$

belongs to $C^\alpha([s, t_1]; X)$ and

$$(2.18) \quad \| \phi \|_{C^\alpha([s,t_1];X)} \leqq ((t_1 - t_0)^\alpha + \alpha^{-1}\nu + 1)M_3(1, 1)[A]_\alpha \| x \|_D \doteq N_{12} \| x \|_D.$$

Proposition 2.1 may be applied to problem (2.3) (with $f = 0$), finding

$$(2.19) \quad \begin{aligned} &\| A(s)w(\cdot) \|_{C^\alpha([s,t_1];X)} + \| w \|_{C^\alpha([s,t_1];X)} + \| w' \|_{C^\alpha([s,t_1];X)} \\ &\leqq \nu \| w(\cdot) \|_{C^\alpha([s,t_1];D)} + \| w' \|_{C^\alpha([s,t_1];X)} \\ &\leqq (\nu + 1)N_8(\alpha)N_{12} \| x \|_D \\ &\doteq N_{13} \| x \|_D. \end{aligned}$$

Therefore (using (2.19) and, respectively, (1.6)–(1.20), (1.7)–(1.8), (1.7), (1.8) again) we have, for any $x \in D$,

$$(2.20) \quad \begin{cases} \text{(i)} \quad \| W(t, s)x \|_{D_{A(t_0)}(\beta,\infty)} \leqq \nu(\beta)M_0N_{13} \| x \|_D, \\ \qquad\qquad t_0 \leqq s \leqq t \leqq t_1, \quad 0 \leqq \beta \leqq 1; \\ \text{(ii)} \quad \| W(t, s)x - W(r, s)x \|_{D_{A(t_0)}(\beta,\infty)} \\ \qquad \leqq \nu(\beta)N_2(t_1 - t_0, \alpha)N_{13}(t - r) \| x \|_D, \\ \qquad\qquad t_0 \leqq s \leqq r \leqq t \leqq t_1, \quad 0 \leqq \beta \leqq \alpha; \\ \qquad \leqq \nu(\beta)N_3(t_1 - t_0, \alpha)N_{13}(t - r)^{\alpha+1-\beta} \| x \|_D, \\ \qquad\qquad t_0 \leqq s \leqq r \leqq t \leqq t_1, \quad \alpha \leqq \beta \leqq 1; \\ \text{(iii)} \quad \left\| \dfrac{\partial}{\partial t} W(t, s)x \right\|_{D_{A(t_0)}(\beta,\infty)} \leqq \nu(\beta)N_2(t_1 - t_0, \alpha)N_{13} \| x \|_D, \\ \qquad\qquad t_0 \leqq s \leqq t \leqq t_1, \quad 0 \leqq \beta \leqq \alpha; \\ \text{(iv)} \quad \left\| \dfrac{\partial}{\partial t} W(t, s)x \Big|_{t=r_2} - \dfrac{\partial}{\partial t} W(t, s)x \Big|_{t=r_1} \right\|_{D_{A(t_0)}(\beta,\infty)} \\ \qquad \leqq \nu(\beta)N_3(t_1 - t_0, \alpha)N_{13}(r_2 - r_1)^{\alpha-\beta} \| x \|_D, \\ \qquad\qquad t_0 \leqq s \leqq r_1 < r_2 \leqq t, \quad 0 \leqq \beta \leqq \alpha. \end{cases}$$

Moreover, by (1.21) (ii), (iii) we have, for $t_0 \leqq s < r < t \leqq t_1$:

$$(2.21) \quad \begin{cases} \text{(i)} \quad \| e^{(t-s)A(s)} \|_{L(D_{A(t_0)}(\theta,\infty), D_{A(t_0)}(\beta,\infty))} \leqq M_4(\theta, \beta)(t - s)^{\theta-\beta}, \\ \qquad\qquad \theta \leqq \beta \leqq 1; \\ \text{(ii)} \quad \| A(s)e^{(t-s)A(s)} \|_{L(D_{A(t_0)}(\theta,\infty), D_{A(t_0)}(\beta,\infty))} \leqq M_4(\theta, \beta)(t - s)^{-1+\theta-\beta}, \\ \qquad\qquad 0 \leqq \beta, \theta \leqq 1; \\ \text{(iii)} \quad \| e^{(t-s)A(s)} - e^{(r-s)A(s)} \|_{L(D_{A(t_0)}(\theta,\infty), D_{A(t_0)}(\beta,\infty))} \\ \qquad \leqq M_4(\theta, \beta) \displaystyle\int_{r-s}^{t-s} \frac{d\sigma}{\sigma^{1-\theta+\beta}}, \quad 0 \leqq \beta, \theta \leqq 1; \end{cases}$$

$$(2.21)\begin{cases}\text{(iv)}\ \| A(s)e^{(t-s)A(s)} - A(s)e^{(r-s)A(s)} \|_{L(D_{A(t_0)}(\theta,\infty),D_{A(t_0)}(\beta,\infty))} \\ \quad \leqq M_4(\theta,\beta)\displaystyle\int_{r-s}^{t-s} \frac{d\sigma}{\sigma^{2-\theta+\beta}}, \quad 0 \leqq \beta, \theta \leqq 1; \\ \text{(v)}\ \| A(s)e^{(t-s)A(s)} - A(s)e^{(r-s)A(s)} \|_{L(D_{A(s)}(\theta+1,\infty),D_{A(t_0)}(\beta,\infty))} \\ \quad \leqq M_4(\theta,\beta)\displaystyle\int_{r-s}^{t-s} \frac{d\sigma}{\sigma^{1-\theta+\beta}}, \quad 0 \leqq \beta, \theta \leqq 1. \end{cases}$$

Now (thanks to (2.9) and (2.16)) (2.10), ..., (2.13) follows from (2.21)(i), ..., (2.21)(iv) and from (2.19), (2.20), (1.10) by interpolation; (2.14) follows from (2.21)(v) and (2.20)(iii), finally (2.15) follows from (2.21)(vi) and (2.20)(iv). □

Propositions 2.1 and 2.5 yield other properties of $G(t,s)$.

PROPOSITION 2.6. *For any* $s \in [t_0, t_1]$ *we have:*

(i) $t \to G(t,s)x \in C^{1+\alpha}([s+\varepsilon, t_1]; X) \cap C^{\alpha}([s+\varepsilon, t_1]; D)$ *for* $x \in X$ *and* $\varepsilon \in]0, t_1 - s[$,

$$\frac{\partial}{\partial t} G(t,s)x = A(t)G(t,s)x \quad \textit{for } s < t \leqq t_1.$$

(ii) $G(r_2, r_1)G(r_1, r_0) = G(r_2, r_0)$ *for* $t_0 \leqq r_0 \leqq r_1 \leqq r_2 \leqq t_1$.

(iii) $\exists \lim_{t\to s^+} G(t,s)x \Leftrightarrow x \in \bar{D}$. *In this case*, $\lim_{t\to s^+} G(t,s)x = x$.

(iv) (a) *If* $0 < \theta \leqq \alpha$ *and* $x \in X$, *then* $G(\cdot, s)x \in C^{\theta}([s, t_1]; X) \Leftrightarrow x \in D_{A(t_0)}(\theta, \infty)$. *In this case*, $G(\cdot, s)x$ *is bounded with values in* $D_{A(t_0)}(\theta, \infty)$.

(b) *If* $\alpha < \theta < 1$ *and* $x \in D_{A(t_0)}(\theta, \infty)$, *then* $G(\cdot, s)x \in C^{\theta}([s, t_1]; X)$.

(c) *If* $x \in D$, *then* $G(\cdot, s)x$ *is Lipschitz continuous with values in* X *and bounded with values in* D. *In particular, it belongs to* $C^{1-\theta}([s, t_1]; D_{A(t_0)}(\theta, \infty))$ *for any* $\theta \in]0, 1[$.

(v) $t \to G(t,s)x \in C^1([s, t_1]; X) \cap C([s, t_1]; D) \Leftrightarrow x \in D, A(s)x \in \bar{D}$.

(vi) *If* $0 < \theta \leqq \alpha$, $t \to G(t,s)x \in C^{1+\theta}([s, t_1]; X) \cap C^{\theta}([s, t_1]; D) \Leftrightarrow x \in D_{A(s)}(\theta + 1, \infty)$.

(vii) *For any* $x \in X$ *the function* $\phi(t) = \int_s^t G(t,\sigma)x\,d\sigma$ *belongs to* $C^1(]s, t_1]; X) \cap C(]s, t_1]; D)$, *and*

$$\phi'(t) = A(t)\int_s^t G(t,\sigma)x\,d\sigma + x. \tag{2.22}$$

There is $c_6 > 0$ *such that*

$$\left\| A(t)\int_s^t G(t,\sigma)x\,d\sigma \right\| \leqq c_6 \| x \|, \qquad t_0 \leqq s < t \leqq t_1, \quad x \in X. \tag{2.23}$$

PROOF. (i) follows from Proposition 2.1. To prove (ii) it is sufficient to show that for any $x \in X$ the function $\phi(t) = G(t, r_1)G(r_1, r_0)x - e^{(t-r_0)A(r_0)}x$ coincides with $W(t, r_0)x$ in the interval $[r_1, t_1]$. This is true because both ϕ and $W(\cdot, r_0)x$ are strict solutions of the problem:

$$\begin{cases} v' = A(t)v(t) + (A(t) - A(r_0))e^{(t-r_0)A(r_0)}x, & r_1 < t \leqq t_1 \\ v(r_1) = G(r_1, r_0)x - e^{(r_1-r_0)A(r_0)}x \end{cases}$$

which has a unique strict solution thanks to Lemma 1.3.

To show (iii) it is sufficient to use (2.9) and to recall that $\lim_{t \to s^+} W(t, s)x = 0$ for any $x \in X$. The statement follows from Proposition 1.2(i) of [Sin].

To show (iv)(a) we use again (2.9): for any $x \in X$, $W(\cdot, s)x$ belongs to $C^\alpha([s, t_1]; X)$, so that the statement follows from Proposition 1.12 of [Sin].

(iv)(b)(c) follow easily from estimates (2.10) and (2.12). Let us show (v) and (vi): the condition $x \in D$ is obviously necessary to get $G(\cdot, s)x \in C([s, t_1]; D)$. Moreover, if $x \in D$ then $W(\cdot, s)x$ belongs to $C^\alpha([s, t_1]; D) \cap C^{1+\alpha}([s, t_1]; X)$, so that $G(\cdot, s)x$ belongs to $C([s, t_1]; D) \cap C^1([s, t_1]; X)$ (resp. $C^\theta([s, t_1]; D) \cap C^{1+\theta}([s, t_1]; X)$, $0 < \theta \leqq \alpha$) if and only if $t \to e^{(t-s)A(s)}x$ does. This happens if and only if $A(s)x$ belongs to $\bar{D}$ (resp. to $D_{A(s)}(\theta, \infty)$): for the proof, see Proposition 1.2(iii) and Proposition 1.12 of [Sin].

Let us show (vii). For $s < t < t + h \leqq t_1$ set $h^{-1}(\phi(t+h) - \phi(t)) = I_{1,h} + I_{2,h} + I_{3,h}$, where

$$I_{1,h} = h^{-1}\left(\int_s^{t+h} W(t+h, \sigma)x\,d\sigma - \int_s^t W(t, \sigma)x\,d\sigma\right),$$

$$I_{2,h} = h^{-1}\left(\int_s^{t+h} (e^{(t+h-\sigma)A(\sigma)} - e^{(t+h-\sigma)A(t)})x\,d\sigma - \int_s^t (e^{(t-\sigma)A(\sigma)} - e^{(t-\sigma)A(t)})x\,d\sigma\right),$$

$$I_{3,h} = h^{-1}\left(\int_s^{t+h} e^{(t+h-\sigma)A(t)}x\,d\sigma - \int_s^t e^{(t-\sigma)A(t)}x\,d\sigma\right).$$

Using (2.2) it is easy to see that

$$\begin{aligned} \lim_{h \to 0^+} I_{1,h} &= \int_s^t \frac{\partial}{\partial t}(W(t, \sigma)x)\,d\sigma \\ &= A(t)\int_s^t W(t, \sigma)x\,d\sigma + \int_s^t (A(t) - A(\sigma)e^{(t-\sigma)A(\sigma)}x\,d\sigma. \end{aligned} \tag{2.24}$$

Moreover we have

$$\left\| I_{2,h} - \int_s^t (A(\sigma)e^{(t-\sigma)A(\sigma)} - A(t)e^{(t-\sigma)A(t)})xd\sigma \right\|$$

$$\leqq \left\| \int_s^t \int_0^1 (A(\sigma)e^{(t-\sigma)A(\sigma)}(e^{\tau hA(\sigma)} - 1)x - A(t)e^{(t-\sigma)A(t)}(e^{\tau hA(t)} - 1)xd\tau d\sigma \right\|$$

$$+ \left\| h^{-1} \int_t^{t+h} (e^{(t+h-\sigma)A(\sigma)} - e^{(t+h-\sigma)A(t)})xd\sigma \right\| .$$

The first integral converges to 0 as $h \to 0$ because

$$\lim_{h\to 0^+} A(\sigma)e^{(t-\sigma)A(\sigma)}(e^{\tau hA(s)} - 1) = \lim_{h\to 0^+} A(t)e^{(t-\sigma)A(t)}(e^{\tau hA(t)} - 1)x = 0,$$

and, by (1.21) (vi), we have

$$\| A(\sigma)e^{(t-\sigma)A(\sigma)}(e^{\tau hA(\sigma)} - 1)x - A(t)e^{(t-\sigma)A(t)}(e^{\tau hA(t)} - 1)x \|$$

$$\leqq \| (A(\sigma)e^{(t+\tau h-\sigma)A(\sigma)} - A(t)e^{(t+\tau h-\sigma)A(t)})x \|$$

$$+ \| (A(\sigma)e^{(t-\sigma)A(\sigma)} - A(t)e^{(t-\sigma)A(t)})x \|$$

$$\leqq 2M_6(0,0)(t-\sigma)^{\alpha-1} \| x \| .$$

Therefore

$$\lim_{h\to 0^+} I_{2,h} = \int_s^t (A(\sigma)e^{(t-\sigma)A(\sigma)} - A(t)e^{(t-\sigma)A(t)})xd\sigma. \tag{2.25}$$

Finally, using the identity

$$I_{3,h} = \Big\{(e^{hA(t)} - 1)e^{(t-s)A(t)}(A(t)-\lambda)^{-1}x$$

$$- \lambda \left(\int_s^{t+h} e^{(t+h-\sigma)A(t)}(A(t)-\lambda)^{-1}xd\sigma - \int_s^t e^{(t-\sigma)A(t)}(A(t)-\lambda)^{-1}xd\sigma \right)\Big\} h^-$$

which holds for any $\lambda \in \rho(A(t))$, we get easily

$$\lim_{h\to 0^+} I_{3,h} = A(t) \int_s^t e^{(t-\sigma)A(t)}xd\sigma + x. \tag{2.26}$$

Therefore ϕ is right differentiable, with

$$\phi'(t) = A(t) \int_s^t G(t,\sigma)xd\sigma + x = A(t)\phi(t) + x \quad \text{for } s < t \leqq t_1.$$

For $t_0 < r_1 < r_2 \leqq t_1$ we have, by (2.9) and (2.3):

$$
\begin{aligned}
&\| A(r_2)\phi(r_2) - A(r_1)\phi(r_1) \| \\
&\leqq \left\| \int_s^{r_1} \left(\frac{\partial}{\partial t} W(t,\sigma)x \,\Big|_{t=r_2} - \frac{\partial}{\partial t} W(t,\sigma)x \,\Big|_{t=r_1} \right) d\sigma \right\| \\
&\quad + \left\| \int_{r_1}^{r_2} \left(\frac{\partial}{\partial t} W(t,\sigma)x \,\Big|_{t=r_1} \right) ds \right\| \\
&\quad + \left\| \int_s^{r_1} \int_{r_1-\sigma}^{r_2-\sigma} [(A(\sigma))^2 e^{\tau A(\sigma)} - (A(r_1))^2 e^{\tau A(r_1)}] x\, d\tau\, d\sigma \right\| \\
&\quad + \left\| \int_{r_1}^{r_2} [A(\sigma) e^{(r_2-\sigma)A(\sigma)} - A(r_2) e^{(r_2-\sigma)A(r_2)}] x\, d\sigma \right\| \\
&\quad + \| e^{(r_2-s)A(r_1)}x - e^{(r_1-s)A(r_1)}x \| + \| e^{(r_2-r_1)A(r_1)}x - e^{(r_2-r_1)A(r_2)}x \| .
\end{aligned}
$$

By (2.4) and (1.3) we get, for any $\varepsilon \in]0, 1[$:

$$
\begin{aligned}
&\left\| \frac{\partial}{\partial t} W(t,\sigma)x \,\Big|_{t=r_2} - \frac{\partial}{\partial t} W(t,\sigma)x \,\Big|_{t=r_1} \right\| \\
&\quad \leqq 2^{\varepsilon} N_9(\alpha) N_1(t_1 - t_0, 1-\alpha, \alpha)(r_2 - r_1)^{(1-\varepsilon)\alpha}(r_1 - \sigma)^{1-\varepsilon\alpha} \| x \| .
\end{aligned}
$$

Therefore, using also (1.21) (v), (iv), (iii) we get

$$
\begin{aligned}
&\| A(r_2)\phi(r_2) - A(r_1)\phi(r_1) \| \\
&\leqq \Big[2^{\varepsilon}(t_1 - t_0)^{\varepsilon\alpha} \varepsilon^{-1} \alpha^{-1} N_9(\alpha) N_1(t_1 - t_0, 1-\alpha, \alpha)(r_2 - r_1)^{(1-\varepsilon)\alpha} \\
&\qquad + \alpha^{-1} N_9(\alpha)(r_2 - r_1)^{\alpha} + M_6(0,0) \left(\int_0^{+\infty} \frac{d\sigma}{\sigma^{1-\alpha}(1+\sigma)} + \alpha^{-1} \right)(r_2 - r_1)^{\alpha} \\
&\qquad + M_1(\log(r_2 - s) - \log(r_1 - s)) + M_5(0,0)(r_2 - r_1)^{\alpha} \Big] \| x \|
\end{aligned}
$$

so that ϕ'_+ is continuous in $]s, t_1]$. Since ϕ is obviously continous it follows $\phi \in C^1(]s, t_1]; X) \cap C(]s, t_1]; D)$.

It remains to show estimate (2.23): by (2.4), (1.21) (v), (i) we have

$$
\begin{aligned}
\| A(t)\phi(t) \| &\leqq \left\| \int_s^t \frac{\partial}{\partial t}(W(t,\sigma)x) d\sigma \right\| \\
&\quad + \left\| \int_s^t (A(\sigma) e^{(t-\sigma)A(\sigma)} - A(t) e^{(t-\sigma)A(t)}) x\, ds \right\| \\
&\quad + \| (e^{(t-s)A(t)} - 1)x \| \\
&\leqq [N_9(\alpha) + M_6(0,0)(t_1 - t_0)^{\alpha}\alpha^{-1} + M_0 + 1] \| x \| . \qquad \square
\end{aligned}
$$

3. The representation formula

In this section we shall show that any solution (in particular, any strong solution) of (1.22) may be written as

$$u(t) = G(t, t_0)x + \int_{t_0}^{t} G(t, s)f(s)ds, \qquad t \geqq t_0. \tag{3.1}$$

Using formula (3.1) and the estimates of Section 2, we will be able to study the regularity properties of the solution.

PROPOSITION 3.1. *Let* $f \in Z_{1-\theta,\theta}(t_0, T; X)$ *and* $x \in D_{A(t_0)}(\theta, \infty)$, $0 < \theta \leqq \alpha$. *Then the function u defined in* (3.1) *is the classical solution of* (1.22) *given by Corollary* 2.3.

PROOF. By Lemma 1.3, it is sufficient to show that u is a classical solution of (1.22) and $\sup_{t_0 < t \leqq T}(t - t_0)^{1-\theta} \| u(t) \|_D < +\infty$. For $t_0 \leqq t \leqq T$ we have, by (1.20) (ii), (2.10), (2.11), (2.23):

$$\begin{aligned}
\| (t - t_0)^{1-\theta}u(t) \|_D &\leqq \nu(t - t_0)^{1-\theta}(\| A(t)u(t) \| + \| u(t) \|) \leqq \nu(t - t_0)^{1-\theta} \\
&\times\Bigg[\| A(t)G(t, t_0)x \| + \left\| \int_{t_0}^{(t_0+t)/2} A(t)G(t, s)(f(s) - f(t))ds \right\| \\
&+ \left\| \int_{(t_0+t)/2}^{t} A(t)G(t, s)(f(s) - f(t))ds \right\| + \left\| A(t) \int_{t_0}^{t} G(t, s)f(t)ds \right\| \\
&+ C_0(0, 0)(\| x \| + (t - t_0)^{\theta}\theta^{-1}|f|_{1-\theta})\Bigg] \\
&\leqq \nu\Bigg[C_0(\theta, 1) \| x \|_{D_{A(t_0)}(\theta,\infty)} \\
&+ C_1(0, 0)\left(\int_0^{1/2} \frac{d\sigma}{(1 + \sigma)\sigma^{1-\theta}} + \log 2\right) |f|_{1-\theta} + C_1(0, 0)2^{\theta-1}[f]_{1-\theta,\theta} \\
&+ C_6|f|_{1-\theta} + C_0(0, 0) \| x \| + C_0(0, 0)(t_1 - t_0)^{\theta}\theta^{-1}|f|_{1-\theta}\Bigg].
\end{aligned}$$

Therefore $\sup_{t_0 < t \leqq t_1}(t - t_0)^{1-\theta} \| u(t) \|_D < +\infty$. Using (iii) of Proposition 2.6 and (2.12) it is easy to see that u belongs to $C([t_0, t_1]; X)$. Let us show that u is a classical solution of (1.22): since

$$\frac{d}{dt} G(t, t_0)x = A(t)G(t, t_0)x,$$

it is sufficient to consider the term $\int_{t_0}^{t} G(t, s)f(s)ds$. For $t_0 < t < t + h \leqq t_1$ set

$$h^{-1}\left[\int_{t_0}^{t+h} G(t+h,s)f(s)ds-\int_{t_0}^{t} G(t,\ s)f(s)ds\right]=I_{1,h}+I_{2,h}+I_{3,h}$$

where

$$I_{1,h}=\int_{t_0}^{t} h^{-1}(G(t+h,s)-G(t,s))(f(s)-f(t))ds$$

converges to $\int_{t_0}^{t} A(t)G(t,s)(f(s)-f(t))ds$, since

$h^{-1}(G(t+h,s)-G(t,s))(f(s)-f(t))$ converges to $A(t)G(t,s)(f(s)-f(t))$

for $s<t$ and its norm is less than

$$\frac{C_1(0,0)}{t-s}\left(\chi_{[t_0,(t_0+t)/2]}(s)\left(\frac{1}{(s-t_0)^{1-\theta}}+\frac{1}{(t-t_0)^{1-\theta}}\right)[f]_{1-\theta}\right.$$
$$\left.+\chi_{[(t_0+t)/2,t]}(s)\frac{(t-s)^{\theta}}{t-t_0}[f]_{1-\theta,\theta}\right),$$

$$I_{2,h}=h^{-1}\left(\int_{t_0}^{t+h} G(t+h,s)f(t)ds-\int_{t_0}^{t} G(t,s)f(t)ds\right)$$

converges to $A(t)\int_{t_0}^{t} G(t,s)f(t)ds+f(t)$ by (vii) of Proposition 2.6,

$$I_{3,h}=h^{-1}\int_{t}^{t+h} G(t+h,s)(f(s)-f(t))ds$$

converges to 0 by (2.10) and the continuity of f.

Therefore the right derivative of u is $A(t)u(t)+f(t)$ for $t>t_0$. Moreover the function $t\to A(t)u(t)$ is easily seen to be continuous in $[t_0,t_1]$: it is sufficient to write, for $t_0<r<t\leqq t_1$,

$$\begin{aligned}A(t)u(t)-A(r)u(r)&=[A(t)G(t,t_0)x-A(r)G(r,t_0)x]\\&+\int_{t_0}^{(t_0+r)/2}(A(t)G(t,s)-A(r)G(r,s))(f(s)-f(r))ds\\&+\int_{(t_0+r)/2}^{r}(A(t)G(t,s)-A(r)G(r,s))(f(s)-f(r))ds\\&+A(t)\int_{t_0}^{t} G(t,s)(f(r)-f(t))ds\\&+\int_{r}^{t} A(t)G(t,s)(f(s)-f(t))ds\\&+\left[A(t)\int_{t_0}^{t} G(t,s)f(t)ds-A(r)\int_{t_0}^{r} G(r,s)f(r)ds\right]\end{aligned}$$

and to use respectively (2.13) (three times), (2.23), (2.11) and (vii) of Proposition 2.6. The continuity of $A(\cdot)u(\cdot)$ implies that u is continuously differentiable in $[t_0, t_1]$ and $u'(t) = A(t)u(t) + f(t)$ for $t_0 \leqq t \leqq t_1$. Then u is a classical solution of (1.22) and, by Lemma 1.3, it coincides with the solution of (1.22) given by Corollary 2.3. □

As a consequence of Proposition 3.1 we get that if f is Hölder continuous in $[t_0, t_1]$ and x belongs to D, $A(t_0)x + f(t_0)$ belongs to $\bar{D}$, then the strict solution of (1.22) (whose existence is stated in Corollary 2.3) is given by formula (3.1). We show now that the same representation formula holds when f is merely continuous.

PROPOSITION 3.2. *Let $f \in C([t_0, t_1]; X)$ and let u be a strict solution of* (1.22). *Then u is given by* (3.1).

PROOF. If u is a strict solution of (1.22), then $u(t_0) = x$ belongs to D. By Proposition 3.1 and Corollary 2.3, the function

$$v(t) = G(t, t_0)x + \int_{t_0}^{t} G(t, s)f(t_0)ds, \qquad t_0 \leqq t < t_1$$

is the strict solution of

$$\begin{cases} v'(t) = A(t)v(t) + f(t_0), & t_0 \leqq t \leqq t_1, \\ v(t_0) = x. \end{cases}$$

Therefore we have only to show that if w is a strict solution of

$$\begin{cases} w'(t) = A(t)w(t) + f(t) - f(t_0), & t_0 \leqq t \leqq t_1 \\ w(t_0) = 0 \end{cases} \tag{3.2}$$

then

$$w(t) = \int_{t_0}^{t} G(t, s)(f(s) - f(t_0))ds.$$

Let w be a strict solution of (3.2), and set

$$w_n(t) = n \int_{t_0}^{t} e^{-n(t-s)} w(s)ds, \qquad n \in \mathbf{N}, \quad t_0 \leqq t \leqq t_1.$$

Then

$$\lim_{n \to \infty} \| w_n - w \|_{C([t_0, t_1]; X)} = 0$$

and, setting $f_n(t) = w_n'(t) - A(t)w_n(t)$, f_n belongs to $C^\alpha([t_0, t_1]; X)$ for any $n \in \mathbf{N}$, so that, by Proposition 3.1, we have

$$w_n(t) = \int_{t_0}^{t} G(t,s) f_n(s)ds, \qquad n \in \mathbf{N}, \quad t_0 \leqq t \leqq t_1. \tag{3.3}$$

On the other hand we have, for $t_0 \leqq t \leqq t_1$,

$$\begin{aligned} f_n(t) &= n \int_0^{t-t_0} e^{-ns}(w'(t-s) - A(t)w(t-s))ds \\ &= n \int_{t_0}^{t} e^{-n(t-s)} f(s)ds + n \int_0^{t-t_0} e^{-ns}(A(t-s) - A(t))w(t-s)ds \end{aligned}$$

so that f_n converges uniformly to f as $n \to \infty$. Letting $n \to \infty$ in (3.3) the proposition is proved. □

From (2.10) and Proposition 3.2 a fundamental *a priori* estimate for the strict solution of (1.22) follows:

$$\| u(t) \| \leqq c_0(0,0) \| x \| + c_0(0,0) \int_{t_0}^{t} \| f(s) \| ds, \quad t_0 \leqq t \leqq t_1. \tag{3.4}$$

Now it is easy to show existence and uniqueness of the strong solution of (1.22) when f is continuous.

Corollary 3.3. *Let $f \in C([t_0, t_1]; X)$ and $x \in \bar{D}$. Then the function u given by* (3.1) *is the unique strong solution of* (1.22).

Proof. Let $f_n \in C^\alpha([t_0, t_1]; X)$ be such that

$$\lim_{n\to\infty} \| f_n - f \|_{C([t_0,t_1];X)} = 0 \quad \text{and} \quad f_n(t_0) = f(t_0) \quad \forall n.$$

Fix $\lambda \in \rho(A(t_0))$ and let $y_n \in D(A(t_0)^2)$ be such that

$$\lim_{n\to\infty} y_n = x + (A(t_0) - \lambda)^{-1} f(t_0).$$

Set $x_n = y_n - (A(t_0) - \lambda)^{-1} f(t_0)$. Then $\lim_{n\to\infty} x_n = x$ and $A(t_0)x_n + f_n(t_0)$ belongs to D. By Proposition 3.2, problem

$$\begin{cases} u_n'(t) = A(t)u_n(t) + f_n(t), & t_0 \leqq t \leqq t_1 \\ u_n(t_0) = x_n \end{cases}$$

has a unique strict solution

$$u_n(t) = G(t, t_0)x_n + \int_{t_0}^{t} G(t, s)f_n(s)ds.$$

Letting $n \to \infty$ and using (3.4) the statement is proved. □

Another consequence of Proposition 3.2 is uniqueness of the classical solution of (1.22) when f is continuous.

COROLLARY 3.4. *Let $f \in C([t_0, t_1]; X)$ and $x \in \bar{D}$. Then any classical solution of (1.22) is also a strong one. In particular, if (1.22) has a classical solution u, then u is given by (3.1).*

PROOF. By Corollary 3.3, the function u given by (3.1) is a strong solution of (1.22); therefore there are $u_n \in C^1([t_0, t_1]; X) \cap C([t_0, t_1]; D)$ such that $u_n \to u$ in $C([t_0, t_1]; X)$ and $f_n = u_n' - A(\cdot)u_n \to f$ in $C([t_0, t_1]; X)$ as $n \to \infty$. Let v be a classical solution of (1.22): then for any $\varepsilon \in]0, t_1 - t_0[$ and $n \in \mathbf{N}$, the function $v_n = v - u_n$ is a strict solution of

$$\begin{cases} v_n'(t) = A(t)v_n(t) + f(t) - f_n(t), & t_0 + \varepsilon \leqq t \leqq t_1 \\ v_n(t_0 + \varepsilon) = v(t_0 + \varepsilon) - u_n(t_0 + \varepsilon) \end{cases}$$

so that

$$\| v(t) - u_n(t) \| \leqq c_0 \left(\| v(t_0 + \varepsilon) - u_n(t_0 + \varepsilon) \| + \int_{t_0 + \varepsilon}^{t} \| f(s) - f_n(s) \| ds \right)$$

by (3.4).

Letting $n \to +\infty$ and $\varepsilon \to 0$ the statement is proved. □

Using the representation formula (3.1) and the estimates of Proposition 2.6 we can prove the regularity properties of the solution of (1.22). In Proposition 2.6 we studied the function $G(\cdot, t_0)x$, and now we consider the function

$$\phi(t) = \int_{t_0}^{t} G(t, s)f(s)ds, \qquad 0 \leqq t_0 \leqq t \leqq t_1.$$

PROPOSITION 3.5. *For $0 < \theta < 1$ there are $N_{14}(\theta), N_{15}(\theta), N_{16}(\theta), N_{17}(\theta) > 0$ depending also on α, T, ν, M_k $(k = 0, \ldots, 6)$, $\| A \|_{C^\alpha([0,T];L(D,X))}$ such that*

(i) *If $f \in L^\infty(t_0, t_1; X)$, then $\phi \in C^{1-\theta}([t_0, t_1]; D_{A(0)}(\theta, \infty))$ for any $\theta \in [0, 1]$, and*

$$\| \phi \|_{C^{1-\theta}([t_0,t_1];D_{A(0)}(\theta,\infty))} \leqq N_{14}(\theta) \| f \|_{L^\infty(t_0,t_1;X)}. \tag{3.5}$$

(ii) *If $f \in C^\theta([t_0, t_1]; X)$ then $\phi \in B([t_0, t_1]; D) \cap C^{\alpha \wedge \theta}([t_0 + \varepsilon, t_1]; D) \cap C^{1 + (\alpha \wedge \theta)}([t_0 + \varepsilon, t_1]; X)$ for any $\varepsilon \in]0, t_1 - t_0[$, and*

$$\sup_{t_0<t<t_1} \| \phi(t) \|_D + \varepsilon(\| \phi \|_{C^{\alpha\wedge\theta}([t_0+\varepsilon,t_1];D)} + \| \phi' \|_{C^{\alpha\wedge\theta}([t_0+\varepsilon,t_1];X)})$$

(3.6)(a)
$$\leqq N_{15}(\theta) \| f \|_{C^\theta([t_0,t_1];X)}.$$

If, in addition, $f(t_0)\in D_{A(0)}(\theta,\infty)$, then $\phi\in C^{\alpha\wedge\theta}([t_0,t_1];D)\cap C^{1+(\alpha\wedge\theta)}([t_0,t_1];X)$ and

$$\| \phi \|_{C^{\alpha\wedge\theta}([t_0,t_1]);D)} + \| \phi' \|_{C^{\alpha\wedge\theta}([t_0,t_1];X)}$$

(3.6)(b)
$$\leqq N_{16}(\theta)(\| f \|_{C^\theta([t_0,t_1];X)} + \| f(t_0) \|_{D_{A(0)}(\theta,\infty)}).$$

(iii) *If $f\in C([t_0,t_1];X)\cap B([t_0,t_1];D_{A(0)}(\theta,\infty))$ then $\phi\in C([t_0,t_1];D)\cap C^1([t_0,t_1];X)$. Moreover ϕ', $A(\cdot)\phi(\cdot)$ belong to $B([t_0,t_1];D_{A(0)}(\alpha\wedge\theta,\infty))$, $A(\cdot)\phi(\cdot)$ belongs to $C^{\alpha\wedge\theta}([t_0,t_1];X)$ and*

$$\sup_{t_0\leqq t<t_1} \| A(t)\phi(t) \|_{D_{A(0)}(\alpha\wedge\theta,\infty)} + \sup_{t_0\leqq t\leqq t_1} \| \phi'(t) \|_{D_{A(0)}(\alpha\wedge\theta,\infty)} + \| A(\cdot)\phi(\cdot) \|_{C^{\alpha\wedge\theta}([t_0,t_1];X)}$$

(3.7)
$$\leqq N_{16}(\theta) \sup_{t_0\leqq t\leqq t_1} \| f(t) \|_{D_{A(0)}(\theta,\infty)}.$$

If, in addition, f belongs to $C([t_0,t_1];D_{A(0)}(\theta))$ and $0<\theta<\alpha$, then u' and $A(\cdot)u(\cdot)$ belong to $C([t_0,t_1];D_{A(0)}(\theta))$.

Proof. (i) Let $t_0\leqq r<t\leqq t_1$. Then by (2.12) and (2.10) we have:

$$\begin{aligned}
&\| \phi(t)-\phi(r) \|_{D_{A(0)}(\theta,\infty)} \\
&\leqq \left\| \int_{t_0}^{r} (G(t,s)-G(r,s))f(s)ds \right\|_{D_{A(0)}(\theta,\infty)} + \left\| \int_r^t G(t,s)f(s)ds \right\|_{D_{A(0)}(\theta,\infty)} \\
&\leqq [C_2(0,\theta)((t-r)^{1-\theta}(t_1-t_0)^\alpha/\alpha+(t-r)^{1-\theta}/\theta(1-\theta)) \\
&\qquad + C_0(0,\theta)(t-r)^{1-\theta}/(1-\theta)][f]_{L^\infty([t_0,t_1];X)}
\end{aligned}$$

and (3.5) follows.

(ii) If f is Hölder continuous, then ϕ is the classical solution of (1.22) with $x=0$ (see Proposition 3.1). By (2.10) and (2.23), (1.20) (ii) we have, for any $t\in[t_0,t_1]$:

$$\begin{aligned}
\| \phi(t) \|_D &\leqq \left\| \int_{t_0}^t G(t,s)(f(s)-f(t))ds \right\|_D + \left\| \int_{t_0}^t G(t,s)f(t)ds \right\|_D \\
&\leqq c_0(0,1)(t_1-t_0)^\theta\theta^{-1}[f]_{C^\theta([t_0,t_1];X)} \\
&\qquad + v(c_0(0,0)(t_1-t_0)+C_6) \sup_{t_0\leqq s\leqq t_1} \| f(s) \|
\end{aligned}$$

which implies (together with (2.4), taking $x=0$) (3.6)(a). In the case that $f(t_0)$ belongs to $D_{A(0)}(\theta,\infty)$, then the regularity properties of ϕ and estimate (3.6)(b) (with $N_{16}(\theta)=N_8(\alpha\wedge\theta)$) follow from Propositions 2.1 and 3.2, taking $s=t_0$ and $x=0$.

(iii) Let $f\in C([t_0,t_1];X)\cap B([t_0,t_1];D_{A(0)}(\theta,\infty))$. Then for $t_0\leqq r<t\leqq t_1$ we have by (2.13) and (2.11):

$$\begin{aligned}
&\|A(t)\phi(t)-A(r)\phi(r)\|\\
&\leqq\left\|\int_{t_0}^{r}(A(t)G(t,s)-A(r)G(r,s))f(s)ds\right\|+\left\|\int_{r}^{t}A(t)G(t,s)f(s)ds\right\|\\
&\leqq\left[C_3(\theta,0)(t_1-t_0)^{\theta}\theta^{-1}(t-r)^{\alpha}+\int_0^{+\infty}\frac{d\sigma}{(1+\sigma)\sigma^{1-\theta}}(t-r)^{\theta}\right.\\
&\qquad\left.+C_1(\theta,0)\theta^{-1}(t-r)^{\theta}\right]\sup_{t_0\leqq s\leqq t_1}\|f\|_{D_{A(t_0)}(\theta,\infty)}
\end{aligned}\tag{3.8}$$

so that $A(\cdot)\phi(\cdot)$ is $\alpha\wedge\theta$-Hölder continuous. To show that it is also bounded with values in $D_{A(t_0)}(\alpha\wedge\theta,\infty)$, we first remark that, due to (2.17)(iii), (2.20)(iii), (1.10), there is $N_{18}(\theta)$ such that

$$\begin{aligned}
&\left\|\frac{\partial}{\partial t}W(t,s)x\right\|_{D_{A(t_0)}(\alpha\wedge\theta,\infty)}\leqq N_{18}(\theta)(t-s)^{-1+(\alpha\wedge\theta)}\|x\|_{D_{A(t_0)}(\theta,\infty)},\\
&\qquad t>s,\quad x\in D_{A(t_0)}(\theta,\infty)
\end{aligned}\tag{3.9}$$

so that, using also (1.21)(v)(iii)(i):

$$\begin{aligned}
&\|A(t)\phi(t)\|_{D_{A(t_0)}(\alpha\wedge\theta,\infty)}\\
&\leqq\left\|\int_{t_0}^{t}\frac{\partial}{\partial t}W(t,s)f(s)ds\right\|_{D_{A(t_0)}(\alpha\wedge\theta,\infty)}\\
&\quad+\left\|\int_{t_0}^{t}(A(s)e^{(t-s)A(s)}-A(t)e^{(t-s)A(t)})f(s)ds\right\|_{D_{A(t_0)}(\alpha\wedge\theta,\infty)}\\
&\quad+\nu(\alpha\wedge\theta)\left\|\int_{t_0}^{t}A(t)e^{(t-s)A(t)}f(s)ds\right\|_{D_{A(t)}(\alpha\wedge\theta,\infty)}\\
&\leqq N_{18}(\alpha\wedge\theta)(t_1-t_0)^{\alpha\wedge\theta}/(\alpha\wedge\theta)\sup_{t_0\leqq s\leqq t_1}\|f(s)\|_{D_{A(t_0)}(\theta,\infty)}\\
&\quad+M_6(\theta,\alpha\wedge\theta)\frac{(t_1-t_0)^{-(\alpha\wedge\theta)+\alpha+\theta}}{\alpha+\theta-\alpha\wedge\theta}\sup_{t_0\leqq s\leqq t_1}\|f(s)\|_{D_{A(t_0)}(\theta,\infty)}\\
&\quad+\nu(\alpha\wedge\theta)(M_4(\theta,0)(t_1-t_0)^{\theta}/\theta\sup_{t_0\leqq s\leqq t_1}\|f(s)\|_{D_{A(t_0)}(\theta,\infty)}
\end{aligned}\tag{3.10}$$

$$+ \sup_{0<\xi\leq 1} \left\| \xi^{1-(\alpha\wedge\theta)} \int_{t_0}^{t} A(t)^2 e^{(t+\xi-s)A(t)} f(s)ds \right\| \Bigg)$$

$$\leq [N_{18}(\alpha\wedge\theta)(t_1-t_0)^{\alpha\wedge\theta}/(\alpha\wedge\theta) + M_6(\theta,\alpha\wedge\theta)(t_1-t_0)^{\alpha\wedge\theta}/(\alpha\wedge\theta)$$

$$+ \nu(\alpha\wedge\theta)(M_4(\theta,0)(t_1-t_0)^{\theta}/\theta + 2^{2-(\alpha\wedge\theta)}M_1/(1-(\alpha\wedge\theta)))]$$

$$\times \sup_{t_0\leq s\leq t_1} \| f(s) \|_{D_{A(t_0)}(\theta,\infty)}.$$

Now it is easy to show that ϕ is differentiable in $[t_0, t_1]$ and $\phi'(t) = A(t)\phi(t) + f(t)$: it is sufficient to write, for $t, t+h \in [t_0, t_1]$,

$$h^{-1}(\phi(t+h)-\phi(t)) = \int_{t_0}^{t} h^{-1}G(t+h,s) - G(t,s))f(s)ds$$

$$+ h^{-1}\int_{t}^{t+h} (G(t+h,s) - G(s+h,s))f(s)ds$$

$$+ h^{-1}\int_{t}^{t+h} (G(s+h,s) - 1)\, f(s)ds$$

$$+ h^{-1}\int_{t}^{t+h} (f(s) - f(t))ds$$

and to use the equality

$$(G(r_2,s) - G(r_1,s))f(s) = \int_{r_1}^{r_2} A(\sigma)G(\sigma,s)f(s)d\sigma$$

and estimate (2.11) in the first three addenda and the continuity of f in the last addendum. Now (3.7) follows from (3.8), (3.10) and from the equality $\phi' = A(\cdot)\phi(\cdot) + f$.

Let finally f belong to $C([t_0, t_1]; D_{A(0)}(\theta))$. Then there are $f_n \in C([t_0, t_1]; D)$ such that

$$\lim_{n\to\infty} \| f_n - f \|_{C([t_0,t_1];D_A(\theta,\infty))} = 0.$$

Setting

$$\phi_n(t) = \int_{t_0}^{t} G(t,s)f_n(s)ds \qquad (t_0 \leq t \leq t_1)$$

then ϕ_n' and $A(\cdot)\phi_n(\cdot)$ belong to $B([t_0, t_1]; D_{A(0)}(\alpha,\infty)) \cap C([t_0, t_1]; X)$ and

$$\lim_{n\to\infty} \sup_{t_0\leq t\leq t_1} \| \phi_n'(t) - \phi'(t) \|_{D_{A(0)}(\theta,\infty)} = 0$$

by (3.8). therefore ϕ' belongs to $C([t_0, t_1]; D_{A(0)}(\theta))$ and so does $A(\cdot)\phi(\cdot)$. □

Proposition 3.5 may be used together with Proposition 2.6 and Corollary 2.3 to study the regularity properties of u, according to the regularity of f and x and to the compatibility conditions between $f(t_0)$ and x. This task is left to the reader.

We conclude this section with some regularity properties of the function $s \to G(t, s)$, which will be used in subsequent papers. We set

$$\Delta = \{(t, s) \in \mathbf{R}^2;\ t_0 \leqq s \leqq t \leqq t_1\}$$

and

$$\Delta_\varepsilon = \{(t, s) \in \mathbf{R}^2;\ t_0 \leqq s \leqq s + \varepsilon \leqq t \leqq t_1\}, \qquad \varepsilon > 0.$$

PROPOSITION 3.6.

(i) *For any* $\varepsilon \in]0, t_1 - t_0[$ *the function* $\Delta_\varepsilon \to L(X)$, $(t, s) \to G(t, s)$ *is* α*-Hölder continuous. In particular, for any* $x \in X$ *and* $t \in]t_0, t_1]$, $G(t, \cdot)x$ *belongs to* $C([t_0, t[; X)$.

(ii) *For any* $x \in \bar{D}$ *and* $t \in]t_0, t_1]$, $G(t, \cdot)x$ *belongs to* $C([t_0, t]; X)$.

(iii) *For any* $\varepsilon \in]0, t_1 - t_0[$ *the function* $(t, s) \to G(t, s)$ *belongs to* $C^{1+\alpha}(\Delta_\varepsilon, L(D, X)) \cap C^\alpha(\Delta_\varepsilon, L(D))$ *and*

$$\frac{\partial}{\partial s} G(t, s) = -G(t, s)A(s) \quad \text{for } t_0 \leqq s < t \leqq t_1.$$

In particular, for any $x \in D$ *and* $t \in]t_0, t_1]$, $G(t, \cdot)x$ *belongs to* $C^1([t_0, t[; X) \cap C([t_0, t[; D)$.

(iv) *For any* $x \in D$ *and* $t \in]t_0, t_1]$ *such that* $A(t)x \in \bar{D}$, $G(t, \cdot)x$ *belongs to* $C^1([t_0, t]; X)$.

PROOF.

(i) Let $\varepsilon \in]0, t_1 - t_0[$, $(r_0, s_0) \in \Delta_\varepsilon$, $x \in X$. Assume $r_1 \geqq r_0$. Then, by (1.21)(iv) and (2.4):

$$\begin{aligned}
&\| G(r_1, s_1)x - G(r_0, s_0)x \| \\
&\leqq \| e^{(r_1 - s_1)A(s_1)}x - e^{(r_1 - s_1)A(s_0)}x \| + \| e^{(r_1 - s_1)A(s_0)}x - e^{(r_0 - s_0)A(s_0)}x \| \\
&\quad + \| W(r_1, s_1)x - W(r_1, s_0)x \| + \| W(r_1, s_0)x - W(r_0, s_0)x \| \\
&\leqq M_5(0, 0)(s_1 - s_0)^\alpha \| x \| + \frac{M_1}{\alpha \varepsilon^\alpha} |(r_1 - s_1) - (r_0 - s_0)|^\alpha \| x \| \\
&\quad + \| W(r_1, s_1)x - W(r_1, s_0)x \| + N_9(\alpha)(r_1 - r_0)^\alpha \| x \| .
\end{aligned}$$

Therefore it is sufficient to estimate $\| W(r_1, s_1)x - W(r_1, s_0)x \|$. For $t \in [\max\{s_0, s_1\}, t_1]$, let $w(t) = W(t, s_1)x - W(t, s_0)x$. If $s_1 \geqq s_0$, w is a classical solution of

$$(3.11) \qquad \begin{cases} w'(t) = A(t)w(t) + (A(t) - A(s_1))e^{(t-s_1)A(s_1)}x - (A(t) \\ \quad - A(s_0))e^{(t-s_0)A(s_0)}x, \quad s_1 \leqq t \leqq t_1, \\ w(s_1) = -W(s_1, s_0)x. \end{cases}$$

Therefore, using first (3.4) and then (2.4), (1.21)(iv)(i) and choosing $\lambda \in \rho(A(t))$ for any t, we get:

$$\begin{aligned}
&\sup_{s_1 \leqq t \leqq r_1} \| w(t) \| \\
&\leqq c_0(0,0) \| W(s_1, s_0)x \| \\
&\quad + c_0(0,0) \left\| \int_{s_1}^{r_1} [(A(\sigma) - A(s_1))e^{(\sigma - s_1)A(s_1)}x - (A(\sigma) - A(s_0))e^{(\sigma - s_0)A(s_0)}x] d\sigma \right\| \\
&\leqq c_0(0,0) N_6(\alpha)(s_1 - s_0)^\alpha \| x \| \\
&\quad + c_0(0,0) \left\| \int_{s_1}^{r_1} (A(\sigma) - A(s_1))(e^{(\sigma - s_1)A(s_1)}x - e^{(\sigma - s_1)A(s_0)}x) d\sigma \right\| \\
&\quad + c_0(0,0) \left\| \int_{s_1}^{r_1} (A(\sigma) - A(s_1))(e^{(\sigma - s_1)A(s_0)}x - e^{(\sigma - s_0)A(s_0)}x) d\sigma \right\| \\
&\quad + c_0(0,0) \left\| (A(s_1) - A(s_0))(A(s_0) - \lambda)^{-1} \int_{s_1}^{r_1} (A(s_0) - \lambda) e^{(\sigma - s_0)A(s_0)} x d\sigma \right\| \\
&\leqq c_0(0,0) \Big[N_9(\alpha) + [A]_\alpha M_5(0,0)(t_1 - t_0)^\alpha/\alpha + [A]_\alpha \nu M_2 \int_0^{+\infty} \frac{d\tau}{(1+\tau)\tau^{1-\alpha}} \\
&\quad + [A]_\alpha \sup_{t_0 \leqq s \leqq t_1} \| (A(s) - \lambda)^{-1} \|_{L(X,D)} (2M_0 + |\lambda|(t_1 - t_0)M_0 \Big] (s_1 - s_0)^\alpha \| x \|.
\end{aligned}$$

If $s_1 \leqq s_0$, we have to replace the initial condition in (3.11) by $w(s_0) = W(s_0, s_1)x$ and the estimate for $\sup_{s_0 < t \leqq r_1} \| w(t) \|$ carries on similarly.

(ii) Follows easily from the equality

$$G(t,s)x - G(t,s_0)x = G(t,s)(1 - G(s,s_0))x \qquad (t_0 \leqq s_0 \leqq s \leqq t)$$

and from Proposition 2.6(iii).

(iii) Let us remark first that for any $x \in D$ we have

$$G(t,s)x - x = \int_s^t G(t,\sigma)A(\sigma)x d\sigma, \qquad t_0 \leqq s \leqq t \leqq t_1. \tag{3.12}$$

Actually, since $\sigma \to A(\sigma)x$ is Hölder continuous, then the function $t \to \int_s^t G(t,\sigma)A(\sigma)x d\sigma$ is the classical solution of

$$\begin{cases} \phi'(t) = A(t)\phi(t) + A(t)x, & s < t \leqq t_1, \\ \phi(s) = 0. \end{cases} \tag{3.13}$$

The same holds for the function $t \to G(t,s)x - x$. Therefore (3.12) holds, and it implies that for $t_0 \leqq t \leqq t_1$ and $s, s+h \in [t_0, t]$ we have

$$h^{-1}(G(t,s)x - G(t,s+h)x) = h^{-1} \int_s^{s+h} G(t,\sigma)A(\sigma)x d\sigma.$$

Point (i) implies that $G(t,\cdot)A(\cdot)$ belongs to $C([t_0, t[; L(D,X))$. Therefore

$$\lim_{h \to 0} \| h^{-1}(G(t,s+h) - G(t,s)) + G(t,s)A(s) \|_{L(D,X)} = 0,$$

so that

$$\frac{\partial}{\partial s} G(t,s) = -G(t,s)A(s).$$

Moreover,

$$(t,s) \to \frac{\partial}{\partial t} G(t,s)$$

belongs to $C^\alpha(\Delta_\varepsilon, L(D,X))$ thanks to (i) and to the α-Hölder continuity of $A(\cdot)$. This fact, together with (3.12) and (3.13), implies that $(t,s) \to A(t)G(t,s)$ belongs to $C^\alpha(\Delta_\varepsilon, L(D,X))$, so that $(t,s) \to G(t,s)$ belongs to $C^\alpha(\Delta_\varepsilon, L(D))$.

(iv) Follows from (3.12) because in this case the function $\sigma \to G(t,\sigma)A(\sigma)x$ is continuous in $[t_0, t]$. □

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